ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, ab = 1$ and $\lambda \geq 0$, then :

$$(a^2 + b^2)(a + b + \lambda) + \frac{4\lambda}{a^2 + b^2} \geq 4(\lambda + 1)$$

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$$\text{Let } Q = \sqrt{\frac{a^2 + b^2}{2}} \text{ and } A = \frac{a + b}{2} \text{ and then :}$$

$$(a^2 + b^2)(a + b + \lambda) + \frac{4\lambda}{a^2 + b^2} \geq 4(\lambda + 1) \Leftrightarrow 2Q^2(2A + \lambda) + \frac{2\lambda}{Q^2} \geq 4(\lambda + 1)$$

$$\Leftrightarrow (2A + \lambda)Q^4 - 2(\lambda + 1)Q^2 + \lambda \overset{(*)}{\geq} 0$$

Now, LHS of $(*)$ is a quadratic polynomial in Q^2 with discriminant $\Delta =$ $4(\lambda + 1)^2 - 4\lambda(2A + \lambda) = 4(\lambda^2 + 2\lambda + 1 - 2A\lambda - \lambda^2) \therefore \Delta = 4\big(1 - 2\lambda(A - 1)\big)$

Now, if $2\lambda(A - 1) \geq 1$, then : $\Delta \leq 0$ and then : LHS of $(*) \geq 0 \Rightarrow (*)$ is true and we now focus on : $2\lambda(A - 1) < 1$ and we have : the bigger zero of LHS of $(*)$

$$= \frac{2(\lambda + 1) + 2\sqrt{1 - 2\lambda(A - 1)}}{2(2A + \lambda)} = \frac{\lambda + 1 + \sqrt{1 - 2\lambda(A - 1)}}{2A + \lambda}$$

$$\leq \frac{\lambda + 1 + \sqrt{1 - 2\lambda(1 - 1)}}{2 + \lambda}\left(\because A = \frac{a + b}{2} \overset{A-G}{\geq} \sqrt{ab} = 1 \Rightarrow A \geq 1 \text{ and } \lambda \geq 0\right) = 1$$

$\therefore$ the bigger zero of LHS of $(*) \leq 1 \rightarrow (1)$ and $Q^2 \overset{Q-A}{\geq} A^2 \geq 1 \overset{via\,(1)}{\geq}$ the bigger zero of LHS of $(*) \Rightarrow (*)$ is true $\therefore (a^2 + b^2)(a + b + \lambda) + \dfrac{4\lambda}{a^2 + b^2} \geq 4(\lambda + 1)$

$$\forall\, a, b > 0 \,\big|\, ab = 1 \wedge \lambda \geq 0, \text{"="} \text{ iff } a = b = 1 \text{ (QED)}$$