ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ *with* $x + y + z = xyz$, then :

$$8\prod_{cyc}\sqrt{x^2+1} \le \left(1 + \frac{1}{3}\sum_{cyc} yz\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$8\prod_{cyc}\sqrt{x^2+1} = 8\prod_{cyc}\sqrt{x^2+\frac{xyz}{x+y+z}} =$$

$$= 8\prod_{cyc}\sqrt{\frac{x}{\sum_{cyc}x}.(x^2+xy+xz+yz)} = 8\prod_{cyc}\sqrt{\frac{x}{\sum_{cyc}x}.(x+y)(x+z)} =$$

$$= 8\sqrt{\frac{xyz}{\left(\sum_{cyc}x\right)^3}}.\prod_{cyc}(x+y) \overset{x+y+z=xyz}{=} 8\sqrt{\frac{\sum_{cyc}x}{\left(\sum_{cyc}x\right)^3}}.\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) - xyz\right) =$$

$$\overset{x+y+z=xyz}{=} \frac{8}{\sum_{cyc}x}.\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right) - \sum_{cyc}x\right) = 8\left(\sum_{cyc}xy - 1\right)$$

$$\Rightarrow 8\prod_{cyc}\sqrt{x^2+1} = 8\left(\sum_{cyc}xy - 1\right) \to (1)$$

$$\text{Now, } xyz = \sum_{cyc}x \overset{A-G}{\ge} 3\sqrt[3]{xyz} \Rightarrow \sqrt[3]{x^2y^2z^2} \ge 3 \therefore \sum_{cyc}xy \overset{A-G}{\ge} 3\sqrt[3]{x^2y^2z^2} \ge 3.3$$

$$\Rightarrow t = \sum_{cyc}xy \ge 9 \to (2)$$

$$\text{Now, } (1) \Rightarrow 8\prod_{cyc}\sqrt{x^2+1} \le \left(1+\frac{1}{3}\sum_{cyc}yz\right)^3 \Leftrightarrow \left(1+\frac{t}{3}\right)^3 \ge 8(t-1)$$

$$\Leftrightarrow (t+3)^3 - 216(t-1) \ge 0 \Leftrightarrow t^3 + 9t^2 - 189t + 243 \ge 0$$

$$\Leftrightarrow (t-9)\left(t^2+15t+3(t-9)\right) \ge 0 \to \text{true} \because t \overset{via\,(2)}{\ge} 9$$

$$\therefore 8\prod_{cyc}\sqrt{x^2+1} \le \left(1+\frac{1}{3}\sum_{cyc}yz\right)^3 \ \forall\, x, y, z > 0 \,|\, x+y+z = xyz,$$

$$'' ='' \text{ iff } x = y = z = \sqrt{3} \text{ (QED)}$$