ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x^2 + y^2 + z^2 = 3$ then:

$$\frac{x^2+1}{y} + \frac{y^2+1}{z} + \frac{z^2+1}{x} \geq 4 + \frac{6}{x+y+z}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3} \overset{AM-HM}{\geq} \frac{3}{x+y+z} \text{ or } \frac{1}{x+y+z} \leq \frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ (1)$$

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq \frac{3(x^2+y^2+z^2)}{x+y+z} \ (2)$$

$$Proof: (x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) \geq 3(x^2+y^2+z^2) or$$

$$\sum \frac{x^3}{y} + \sum \frac{xy^2}{z} \geq 2\sum x^2 \ or$$

$$\sum \frac{x^4}{yx} + \sum \frac{x^2y^2}{zx} \geq 2\sum x^2 \ or$$

$$\frac{(\sum x^2)^2}{\sum xy} + \frac{(\sum xy)^2}{\sum xy} \geq 2\sum x^2 \ (Bergstrom)$$

$$or \left(\sum x^2\right)^2 + \left(\sum xy\right)^2 - 2\left(\sum xy\right)\left(\sum x^2\right) \geq 0 \ or$$

$$\left(\sum x^2 - \sum xy\right)^2 \geq 0, proof\ complete$$

$$(x+y+z)^2 \overset{CBS}{\leq} 3(x^2+y^2+z^2) \text{ or } (x+y+z) \leq \sqrt{3(x^2+y^2+z^2)} \ (3)$$

$$\frac{x^2+1}{y} + \frac{y^2+1}{z} + \frac{z^2+1}{x} - \frac{6}{x+y+z} = \left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) + \left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right) - \left(\frac{6}{x+y+z}\right)$$

$$\overset{(1)\&(2)}{\geq} \frac{3(x^2+y^2+z^2)}{x+y+z} + \left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right) - \frac{6}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) =$$

$$= \frac{3(x^2+y^2+z^2)}{x+y+z} + \frac{3}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\overset{Bergstrom}{\geq} \frac{3(x^2+y^2+z^2)}{x+y+z} + \frac{1}{3}\frac{(1+1+1)^2}{x+y+z} \overset{(3)}{\geq} \frac{3(x^2+y^2+z^2)}{\sqrt{3(x^2+y^2+z^2)}} + \frac{1}{3}\frac{9}{\sqrt{3(x^2+y^2+z^2)}}$$

$$= \frac{3.3}{\sqrt{3.3}} + \frac{3}{\sqrt{3.3}} (since\ x^2+y^2+z^2 = 3) = 3 + \frac{3}{3} = 4$$

$$or \frac{x^2+1}{y} + \frac{y^2+1}{z} + \frac{z^2+1}{x} \geq 4 + \frac{6}{x+y+z}$$

$$Equality\ holds\ for\ x = y = z = 1$$