ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, abc=1$ then:

$$\sum\frac{a^3+b^3}{a^2+\lambda ab+b^2}\geq\frac{6}{\lambda+2}$$

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$$a^2+\lambda ab+b^2\overset{AM-GM}{\leq}a^2+b^2+\frac{\lambda(a^2+b^2)}{2}=\frac{(a^2+b^2)(\lambda+2)}{2}\quad(1)$$

$$a^3+b^3=(a+b)(a^2+b^2-ab)$$

$$Now\frac{a^3+b^3}{a^2+\lambda ab+b^2}\overset{(1)}{\geq}\frac{(a+b)(a^2+b^2-ab)}{\frac{(a^2+b^2)(\lambda+2)}{2}}=$$

$$=\frac{2}{\lambda+2}(a+b)\left(\frac{a^2+b^2-ab}{a^2+b^2}\right)=\frac{2}{\lambda+2}(a+b)\left(1-\frac{ab}{a^2+b^2}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{2}{\lambda+2}(a+b)\left(1-\frac{ab}{2ab}\right)=\frac{2}{\lambda+2}(a+b)\left(1-\frac{1}{2}\right)=\frac{a+b}{\lambda+2}\ (2)$$

$$\sum\frac{a^3+b^3}{a^2+\lambda ab+b^2}\overset{(2)}{\geq}\sum\frac{a+b}{\lambda+2}\overset{AM-GM}{\geq}\frac{6}{\lambda+2}(a^2b^2c^2)^{\frac{1}{6}}=\frac{6}{\lambda+2}$$

$$Equality\ for\ a=b=c=1$$