ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, n\in\mathbb{N}$ then:

$$\sum_{cyc}\frac{a^{n+1}}{b^n+c^n}\ge\frac{a+b+c}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by proposer

For $n=0$: $\frac{a+b+c}{2}=\frac{a+b+c}{2}$.

For $n\in\mathbf{N}^*$:

$\left(a^{n+1},b^{n+1},c^{n+1}\right)$, $\left(\frac{1}{b^n+c^n},\frac{1}{c^n+a^n},\frac{1}{a^n+b^n}\right)$ are same ordered.

By Chebyshev :

$$LHS=\sum\frac{a^{n+1}}{b^n+c^n}\overset{Chebyshev}{\ge}\frac{1}{3}\sum a^{n+1}\sum\frac{1}{b^n+c^n}\overset{CS}{\ge}\frac{1}{3}\sum a^{n+1}\frac{9}{\sum\left(b^n+c^n\right)}=\frac{1}{3}\sum a^{n+1}\frac{9}{2\sum a^n}=$$

$$=\frac{3\sum a^{n+1}}{2\sum a^n}\overset{(1)}{\ge}\frac{\sum a}{2}=RHS,$$

$$(1)\Leftrightarrow\frac{3\sum a^{n+1}}{2\sum a^n}\ge\frac{\sum a}{2}\Leftrightarrow 3\sum a^{n+1}\ge\sum a^n\sum a.$$

Equality holds for $a=b=c$.

Solution 2 by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^{n+1}}{b^n+c^n}\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^n}{b^{n-1}+c^{n-1}}\overset{LEHMER}{\ge}\sum_{cyc}\frac{a^{n-1}}{b^{n-2}+c^{n-2}}\overset{LEHMER}{\ge}$$

$$\ge\sum_{cyc}\frac{a^2}{b+c}\overset{BERGSTROM}{\ge}\frac{(a+b+c)^2}{b+c+c+a+a+b}=\frac{a+b+c}{2}$$

Equality holds for $a=b=c$.