ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, $n\in N$ then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{b^{n+2}+c^{n+2}}{a^3(b^n+c^n)}\ge 6$$

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Solution by Tapas Das-India

$$\frac{(b^{n+2}+c^{n+2})}{2}\overset{CBS}{\ge}\frac{b^n+c^n}{2}.\frac{b^2+c^2}{2}\,or$$
$$(b^{n+2}+c^{n+2})\ge\frac{1}{2}(b^2+c^2)(b^n+c^n)\overset{AM-GM}{\ge}$$
$$\ge\frac{1}{2}2bc(b^n+c^n)=bc(b^n+c^n)\ (1)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{b^{n+2}+c^{n+2}}{a^3(b^n+c^n)}\overset{(1)}{\ge}\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{bc(b^n+c^n)}{a^3(b^n+c^n)}=$$

$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{bc}{a^3}=$$

$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\overset{AM-GM}{\ge}$$

$$\ge 6\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\frac{bc}{a^3}\cdot\frac{ca}{b^3}\cdot\frac{ab}{c^3}\right)^{\frac{1}{3}}=6\left(\frac{1}{a^2b^2c^2}\right)^{\frac{1}{3}}=6(as\ abc=1)$$

$$Equality\ a=b=c$$