ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x, y > 0 \text{ and } \lambda \leq \frac{5}{4}, \text{then :}$$

$$\frac{x^4 + y^4}{(x+y)^4} + \lambda.\frac{\sqrt{xy}}{x+y} \geq \frac{4\lambda + 1}{8}$$

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$$\frac{x^4 + y^4}{(x+y)^4} + \lambda.\frac{\sqrt{xy}}{x+y} \overset{G-H}{\geq} \frac{x^4 + y^4}{(x+y)^4} + \lambda.\frac{2xy}{(x+y)^2} \overset{?}{\geq} \frac{4\lambda + 1}{8}$$

$$\Leftrightarrow \frac{x^4 + y^4}{(x+y)^4} - \frac{1}{8} \overset{?}{\geq} \lambda.\left(\frac{1}{2} - \frac{2xy}{(x+y)^2}\right)$$

$$\Leftrightarrow \frac{7x^4 + 7y^4 - 4x^3y - 4xy^3 - 6x^2y^2}{8(x+y)^4} \overset{?}{\geq} \lambda.\frac{(x-y)^2}{2(x+y)^2}$$

$$\Leftrightarrow \frac{(7x^4 + 7y^4 - 14x^2y^2) - (4x^3y + 4xy^3 - 8x^2y^2)}{8(x+y)^4} \overset{?}{\geq} \lambda.\frac{(x-y)^2}{2(x+y)^2}$$

$$\Leftrightarrow \frac{7(x-y)^2(x+y)^2 - 4xy(x-y)^2}{4(x+y)^4} \underset{(*)}{\overset{?}{\geq}} \lambda.\frac{(x-y)^2}{(x+y)^2} \text{ and } \because \lambda \leq \frac{5}{4} \therefore \text{in order}$$

$$\text{to prove } (*), \text{it sufficesto prove : } \frac{7(x-y)^2(x+y)^2 - 4xy(x-y)^2}{4(x+y)^4} \geq \frac{5}{4}.\frac{(x-y)^2}{(x+y)^2}$$

$$\Leftrightarrow \frac{(x-y)^2}{(x+y)^2}.\left(\frac{7(x+y)^2 - 4xy}{(x+y)^2} - 5\right) \geq 0 \Leftrightarrow \frac{(x-y)^2}{(x+y)^2}.\left(\frac{2(x+y)^2 - 4xy}{(x+y)^2}\right) \geq 0$$

$$\Leftrightarrow \frac{(x-y)^2}{(x+y)^2}.\frac{x^2 + y^2}{(x+y)^2} \geq 0 \rightarrow \text{true}$$

$$\therefore \frac{x^4 + y^4}{(x+y)^4} + \lambda.\frac{\sqrt{xy}}{x+y} \geq \frac{4\lambda + 1}{8} \; \forall \, x, y > 0 \text{ and } \lambda \leq \frac{5}{4}, '' ='' \text{ iff } x = y \text{ (QED)}$$