ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ with $ab \geq \frac{1}{3}$ and $\lambda \geq \frac{1}{3}$, then :

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\leq\frac{6}{3\lambda+1}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\leq\frac{6}{3\lambda+1}$$

$$\Leftrightarrow (3\lambda+1)(a^2+b^2+2\lambda)\leq 6(a^2b^2+\lambda a^2+\lambda b^2+\lambda^2)$$

$$\Leftrightarrow 3\lambda(a^2+b^2)+a^2+b^2+2\lambda\leq 6a^2b^2+6\lambda(a^2+b^2)$$

$$\Leftrightarrow a^2+b^2+2\lambda-6a^2b^2-3\lambda(a^2+b^2)\overset{(*)}{\leq} 0$$

$$\text{Now}, \because ab\geq\frac{1}{3}\therefore \text{LHS of } (*)\leq a^2+b^2+6\lambda ab-2ab-3\lambda(a^2+b^2)$$

$$=(a-b)^2-3\lambda(a-b)^2=(1-3\lambda)(a-b)^2\overset{\lambda\geq\frac{1}{3}}{\leq} 0\Rightarrow (*)\text{ is true}$$

$$\therefore\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\leq\frac{6}{3\lambda+1}\ \forall\, a,b\in\mathbb{R}\text{ with } ab\geq\frac{1}{3}\text{ and } \lambda\geq\frac{1}{3}\ (\text{QED})$$