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If $a, \mathrm{b}, \mathrm{c} > 0$ *with* $a + \mathrm{b} + \mathrm{c} = 3a\mathrm{bc}$ *and* $\lambda \geq 3$, then :

$$\sum_{cyc} \frac{\mathrm{bc}}{\lambda a + \mathrm{bc} + a\mathrm{bc}} \geq \frac{3}{\lambda + 2}$$

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$$\sum_{cyc} \frac{\mathrm{bc}}{\lambda a + \mathrm{bc} + a\mathrm{bc}} = \sum_{cyc} \frac{\mathrm{b}^2\mathrm{c}^2}{\lambda a\mathrm{bc} + \mathrm{b}^2\mathrm{c}^2 + a\mathrm{bc}.\mathrm{bc}} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\mathrm{b}\right)^2}{3\lambda a\mathrm{bc} + \sum_{cyc} a^2\mathrm{b}^2 + a\mathrm{bc}\sum_{cyc} a\mathrm{b}} \overset{\sum_{cyc} a = 3a\mathrm{bc}}{=} \frac{\left(\sum_{cyc} a\mathrm{b}\right)^2}{3\lambda a\mathrm{bc}.\sqrt{\frac{3a\mathrm{bc}}{\sum_{cyc} a}} + \sum_{cyc} a^2\mathrm{b}^2 + a\mathrm{bc}\left(\sum_{cyc} a\mathrm{b}\right).\sqrt{\frac{\sum_{cyc} a}{3a\mathrm{bc}}}}$$

$$= \frac{\left(\sum_{cyc} a\mathrm{b}\right)^2.\sqrt{3a\mathrm{bc}\sum_{cyc} a}}{3\lambda a\mathrm{bc}.3a\mathrm{bc} + \left(\sum_{cyc} a^2\mathrm{b}^2\right).\sqrt{3a\mathrm{bc}\sum_{cyc} a} + a\mathrm{bc}\left(\sum_{cyc} a\mathrm{b}\right).\left(\sum_{cyc} a\right)} \overset{?}{\geq} \frac{3}{\lambda + 2}$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} a\mathrm{b}\right)^2.\sqrt{3a\mathrm{bc}\sum_{cyc} a} - 27a^2\mathrm{b}^2\mathrm{c}^2\right) + 2\left(\sum_{cyc} a\mathrm{b}\right)^2.\sqrt{3a\mathrm{bc}\sum_{cyc} a}$$

$$\overset{?}{\geq} 3\left(\sum_{cyc} a^2\mathrm{b}^2\right).\sqrt{3a\mathrm{bc}\sum_{cyc} a} + 3a\mathrm{bc}\left(\sum_{cyc} a\mathrm{b}\right).\left(\sum_{cyc} a\right)$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} a\mathrm{b}\right)^2 - \frac{27a^2\mathrm{b}^2\mathrm{c}^2}{\sqrt{3a\mathrm{bc}\sum_{cyc} a}}\right) + 2\left(\sum_{cyc} a\mathrm{b}\right)^2 \boxed{\overset{?}{\geq}_{(*)}}$$

$$3\left(\sum_{cyc} a^2\mathrm{b}^2\right) + \left(\sum_{cyc} a\mathrm{b}\right).\sqrt{3a\mathrm{bc}\sum_{cyc} a}$$

$$\text{Now, }\left(\sum_{cyc} a\mathrm{b}\right)^2.\sqrt{3a\mathrm{bc}\sum_{cyc} a} \overset{?}{\geq} 27a^2\mathrm{b}^2\mathrm{c}^2 \Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} a\mathrm{b}\right)^4 \overset{?}{\geq} 243a^3\mathrm{b}^3\mathrm{c}^3$$

$$\rightarrow \text{true} \because \left(\sum_{cyc} a\right)\left(\sum_{cyc} a\mathrm{b}\right) \overset{A-G}{\geq} 9a\mathrm{bc} \text{ and } \left(\sum_{cyc} a\mathrm{b}\right)^3 \overset{A-G}{\geq} 27a^2\mathrm{b}^2\mathrm{c}^2$$

$$\therefore \left(\sum_{cyc} a\mathrm{b}\right)^2.\sqrt{3a\mathrm{bc}\sum_{cyc} a} - 27a^2\mathrm{b}^2\mathrm{c}^2 \geq 0 \Rightarrow \left(\sum_{cyc} a\mathrm{b}\right)^2 - \frac{27a^2\mathrm{b}^2\mathrm{c}^2}{\sqrt{3a\mathrm{bc}\sum_{cyc} a}} \geq 0 \text{ and}$$

$$\because \lambda \geq 3 \therefore \text{LHS of } (*) - \text{RHS of } (*) \geq 3\left(\left(\sum_{\text{cyc}} ab\right)^2 - \frac{27a^2b^2c^2}{\sqrt{3abc\sum_{\text{cyc}} a}}\right)$$

$$+2\left(\sum_{\text{cyc}} ab\right)^2 - 3\left(\sum_{\text{cyc}} a^2b^2\right) - \left(\sum_{\text{cyc}} ab\right).\sqrt{3abc\sum_{\text{cyc}} a}$$

$$= 2\sum_{\text{cyc}} a^2b^2 + 10abc\sum_{\text{cyc}} a - \frac{81a^2b^2c^2}{\sqrt{3abc\sum_{\text{cyc}} a}} - \left(\sum_{\text{cyc}} ab\right).\sqrt{3abc\sum_{\text{cyc}} a} \overset{?}{\geq} 0$$

$$\Leftrightarrow 2\sum_{\text{cyc}} a^2b^2 + 10abc\sum_{\text{cyc}} a \overset{?}{\geq} \frac{81a^2b^2c^2 + 3abc(\sum_{\text{cyc}} a)(\sum_{\text{cyc}} ab)}{\sqrt{3abc\sum_{\text{cyc}} a}}$$

$$\Leftrightarrow 3abc\left(\sum_{\text{cyc}} a\right)\left(2\sum_{\text{cyc}} a^2b^2 + 10abc\sum_{\text{cyc}} a\right)^2 \boxed{\overset{?}{\underset{(**)}{\geq}}}$$

$$\left(81a^2b^2c^2 + 3abc\left(\sum_{\text{cyc}} a\right)\left(\sum_{\text{cyc}} ab\right)\right)^2$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ *and* $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of *a* triangle with semiperimeter, circumradius *and* inradius $= s, R, r$ (*say*);

$$\text{so } 2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2s \to (2) \text{ and such substitutions} \Rightarrow \sum_{\text{cyc}} ab = \sum_{\text{cyc}} (s - x)(s - y)$$

$$\Rightarrow \sum_{\text{cyc}} ab = 4Rr + r^2 \to (3), \text{and } \sum_{\text{cyc}} a^2b^2 = \left(\sum_{\text{cyc}} ab\right)^2 - 2abc\left(\sum_{\text{cyc}} a\right)$$

$$\overset{\text{via } (1),(2) \text{ and } (3)}{=} (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{\text{cyc}} a^2b^2 = r^2((4R + r)^2 - 2s^2) \to (4)$$

and via $(1), (2), (3)$ *and* $(4), (**)$

$$\Leftrightarrow 3r^2s^2(2r^2((4R + r)^2 - 2s^2) + 10r^2s^2)^2 \geq \left(81r^4s^2 + 3r^2s^2(4Rr + r^2)\right)^2$$

$$\Leftrightarrow (4R + r)^4 + (84R^2 - 120Rr - 582r^2)s^2 + 9s^4 \boxed{\overset{(***)}{\geq}} 0$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (4R + r)^4 + (84R^2 - 120Rr - 582r^2)s^2$$

$$+9(16Rr - 5r^2)s^2 \overset{?}{\geq} 0 \Leftrightarrow (84R^2 + 24Rr - 627r^2)s^2 + (4R + r)^4 \boxed{\overset{(****)}{\geq}} 0$$

$\boxed{\text{Case 1}}$ $84R^2 + 24Rr - 627r^2 \geq 0$ *and then* : LHS of $(****) \geq (4R + r)^4 > 0$

$$\Rightarrow (****) \text{ is true (strict inequality)}$$

$\boxed{Case\ 2}$ $84R^2 + 24Rr - 627r^2 < 0$ *and* then : LHS of $(****) \overset{\text{Gerretsen}}{\geq}$

$$(84R^2 + 24Rr - 627r^2)(4R^2 + 4Rr + 3r^2) + (4R + r)^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow 148t^4 + 172t^3 - 516t^2 - 605t - 470 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(148t^3 + 468t^2 + 420t + 235) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \text{ is true}$$

$\therefore$ combining both *cases*, $(****) \Rightarrow (***)$ is true for *all* triangles $\therefore (**) \Rightarrow (*)$ is true $\Rightarrow \sum_{cyc} \frac{bc}{\lambda a + bc + abc} \geq \frac{3}{\lambda + 2}\ \forall\, a, b, c > 0 \mid a + b + c = 3abc$ *and* $\lambda \geq 3$, " = " iff $a = b = c = 1$ (QED)