ROMANIAN MATHEMATICAL MAGAZINE

**If $a, b, c > 0$ with $abc = 1$ and $n \in \mathbb{N}$, then :**

$$\sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} \le 1$$

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$$\sum_{cyc} \frac{1}{1 + b^3 + c^3} \le \sum_{cyc} \frac{1}{1 + bc(b + c)} \overset{abc=1}{=} \sum_{cyc} \frac{1}{1 + \frac{b+c}{a}} = \sum_{cyc} \frac{a}{a + b + c}$$

$$\therefore \sum_{cyc} \frac{1}{1 + b^3 + c^3} \le 1 \to (1)$$

$\boxed{Case\ 1}$ $n = 0$ and then : $\displaystyle\sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} = \sum_{cyc} \frac{1}{1 + b^3 + c^3} \overset{via\ (1)}{\le} 1$

$\boxed{Case\ 2}$ $n \in \mathbb{N}^*$ and then : $f(x) = x^n \Rightarrow f''(x) = n(n-1)x^{n-2} \ge 0$

$$\Rightarrow f(x) \text{ is convex} \Rightarrow \frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3} = \frac{1.a^n + b^3.b^n + c^3.c^n}{1 + b^3 + c^3} \overset{\text{Weighted Jensen}}{\ge}$$

$$\left(\frac{1.a + b^3.b + c^3.c}{1 + b^3 + c^3}\right)^n \Rightarrow \sqrt[n]{\frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3}} \ge \frac{a + b^4 + c^4}{1 + b^3 + c^3} \overset{abc=1}{=} \frac{\frac{1}{bc} + \frac{b^6}{b^2} + \frac{c^6}{c^2}}{1 + b^3 + c^3}$$

$$\overset{\text{Bergstrom}}{\ge} \frac{(1 + b^3 + c^3)^2}{(b^2 + c^2 + bc)(1 + b^3 + c^3)} \overset{A-G}{\ge} \frac{1 + b^3 + c^3}{b^2 + c^2 + \frac{b^2+c^2}{2}} \overset{\text{Power-Mean Inequality}}{\ge} \frac{1 + 2\left(\frac{b^2+c^2}{2}\right)^{\frac{3}{2}}}{\frac{3}{2}(b^2 + c^2)}$$

$$= 1 + \frac{1 + 2t^3 - 3t^2}{3t^2} \left(t = \sqrt{\frac{b^2 + c^2}{2}}\right) = 1 + \frac{(2t+1)(t-1)^2}{3t^2} \ge 1$$

$$\therefore \sqrt[n]{\frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3}} \ge 1 \Rightarrow \frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3} \ge 1 \ (\because n \in \mathbb{N}^* \Rightarrow n > 0)$$

$$\Rightarrow \frac{1}{a^n + b^{n+3} + c^{n+3}} \le \frac{1}{1 + b^3 + c^3} \text{ and analogs} \Rightarrow \sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}}$$

$$\le \sum_{cyc} \frac{1}{1 + b^3 + c^3} \overset{via\ (1)}{\le} 1 \therefore \text{combining both cases}, \sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} \le 1$$

$$\forall\, a, b, c > 0 \,\big|\, abc = 1 \wedge n \in \mathbb{N}, \text{''}=\text{''} \text{ iff } a = b = c = 1 \text{ (QED)}$$