If $a, \mathrm{b}, \mathrm{c}, \lambda > 0$ and $a\mathrm{b} + \mathrm{bc} + \mathrm{c}a = \lambda$, then :

$$\frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}} \geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}} \geq 2 \Leftrightarrow \frac{\sum_{cyc}\sqrt{a^2+ab+bc+ca}}{\sum_{cyc}\sqrt{ab}} \geq 2$$

$$\Leftrightarrow \boxed{\sum_{cyc}\sqrt{(a+b)(c+a)} \overset{(*)}{\geq} 2\sum_{cyc}\sqrt{ab}}$$

Assigning $b+c=x', c+a=y', a+b=z' \Rightarrow x'+y'-z'=2c>0, y'+z'-x'$ $=2a>0$ and $z'+x'-y'=2b>0 \Rightarrow x'+y'>z', y'+z'>x', z'+x'>y'$ $\Rightarrow x', y', z'$ form sides of a triangle $\Rightarrow \sqrt{b+c}=\sqrt{x'}=x, \sqrt{c+a}=\sqrt{y'}=y,$ $\sqrt{a+b}=\sqrt{z'}=z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{Now, } b+c=x^2, c+a=y^2, a+b=z^2 \Rightarrow \sum_{cyc} a = \frac{x^2+y^2+z^2}{2}$$

$$\Rightarrow a=\frac{y^2+z^2-x^2}{2}, b=\frac{z^2+x^2-y^2}{2}, c=\frac{x^2+y^2-z^2}{2}$$

$$\therefore \text{ using such transformations, } \sum_{cyc}\sqrt{(a+b)(c+a)} \geq 2\sum_{cyc}\sqrt{ab}$$

$$\Leftrightarrow \sum_{cyc} yz \geq 2\sum_{cyc}\sqrt{\frac{y^2+z^2-x^2}{2}\cdot\frac{z^2+x^2-y^2}{2}}$$

$$\Leftrightarrow \boxed{\begin{array}{c}\sum_{cyc} x^2y^2 + 2xyz\sum_{cyc} x \overset{(**)}{\geq} \sum_{cyc}(y^2+z^2-x^2)(z^2+x^2-y^2) \\ +2\sum_{cyc}\left((y^2+z^2-x^2).\sqrt{(z^2+x^2-y^2)(x^2+y^2-z^2)}\right)\end{array}}$$

$$\text{Now, RHS of } (**) \overset{A-G}{\leq} 2\sum_{cyc}\left((y^2+z^2-x^2).\frac{(z^2+x^2-y^2)+(x^2+y^2-z^2)}{2}\right)$$

$$= 2\sum_{cyc} x^2(y^2+z^2-x^2) = 4\sum_{cyc} x^2y^2 - 2\sum_{cyc} x^4$$

$$\therefore \text{ LHS of } (**) \leq -\sum_{cyc} x^4 + 2\sum_{cyc} x^2y^2 + 4\sum_{cyc} x^2y^2 - 2\sum_{cyc} x^4 \overset{?}{\leq}$$

$$\sum_{cyc} x^2y^2 + 2xyz\sum_{cyc} x \Leftrightarrow 3\left(2\sum_{cyc} x^2y^2 - 16r^2s^2\right) - 5\sum_{cyc} x^2y^2 + 2.4Rrs.2s \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\sum_{cyc} xy\right)^2 - 16Rrs^2 - 48r^2s^2 + 16Rrs^2 \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc} xy \overset{?}{\geq} 4\sqrt{3}rs$$

$$\rightarrow \text{true via Gordon} \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{\sum_{cyc}\sqrt{a^2+\lambda}}{\sum_{cyc}\sqrt{ab}} \geq 2\ \forall\, a, b, c, \lambda > 0$$

$$\text{and } ab + bc + ca = \lambda, '' ='' \text{ iff } a = b = c = \sqrt{\frac{\lambda}{3}}\ (\text{QED})$$