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If $a,b,c>0, abc=1$ then:

$$\sum \frac{1}{b+c}+\frac{1}{2}\ge \frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)}$$

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$$(a+b)(b+c)(c+a)=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc \ (1)$$

$$a^2(b+c)+a(b+c)=(b+c)(a^2+a)\overset{AM-GM}{\ge} 2\sqrt{bc}\,2\sqrt{a^3}=4a\sqrt{abc}\overset{abc=1}{=}4a$$

$$\sum\left(a^2(b+c)+a(b+c)\right)=\sum(a^2(b+c)+ab+ac)\ge 4\sum a \ (2)$$

$$We\ need\ to\ show\ \sum \frac{1}{b+c}+\frac{1}{2}\ge \frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)}\ or$$

$$\frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)}-\sum\frac{1}{b+c}\le\frac{1}{2}$$

$$\frac{(a+b+c)^2+2(a+b+c)+1-(\sum(a+b)(a+c))}{\prod(a+b)}\le\frac{1}{2}$$

$$2\left[\left(\sum a^2\right)+2\left(\sum ab\right)+2(a+b+c)+1-\left(\sum(a^2+ab+bc+ca)\right)\right]\overset{(1)}{\le}$$
$$\le a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$$

$$or\ 4\left(\sum a\right)-2\left(\sum ab\right)+2\overset{abc=1}{\le}\sum a^2(b+c)+2$$

$$or\ 4\left(\sum a\right)\le\sum a^2(b+c)+2\left(\sum ab\right)\ or$$

$$4\left(\sum a\right)\le\sum(a^2(b+c)+ab+ac)\ True\ from\ (2)$$

$$Equality\ holds\ for\ a=b=c=1$$