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If $a,b,c>0$ with $a+b+c=ab+bc+ca$ and $\lambda\ge 2$, then :

$$\sum_{cyc}\frac{a}{\lambda a+bc}\le\frac{3}{\lambda+1}$$

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$$\sum_{cyc}\frac{a}{\lambda a+bc}\le\frac{3}{\lambda+1}\Leftrightarrow\frac{1}{\lambda}.\sum_{cyc}\frac{\lambda a+bc-bc}{\lambda a+bc}\le\frac{3}{\lambda+1}$$

$$\Leftrightarrow\frac{1}{\lambda}.\sum_{cyc}\frac{bc}{\lambda a+bc}\ge\frac{3}{\lambda}-\frac{3}{\lambda+1}\Leftrightarrow\sum_{cyc}\frac{bc}{\lambda a+bc}\overset{(*)}{\ge}\frac{3}{\lambda+1}$$

$$\text{Now, }\sum_{cyc}\frac{bc}{\lambda a+bc}=\sum_{cyc}\frac{b^2c^2}{\lambda abc+b^2c^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}ab\right)^2}{3\lambda abc+\sum_{cyc}a^2b^2}\overset{?}{\ge}\frac{3}{\lambda+1}$$

$$\Leftrightarrow(\lambda+1)\left(\sum_{cyc}a^2b^2\right)+2(\lambda+1)abc\left(\sum_{cyc}a\right)\overset{?}{\ge}9\lambda abc+3\sum_{cyc}a^2b^2$$

$$\Leftrightarrow(\lambda-2)\left(\sum_{cyc}a^2b^2\right).\frac{\sum_{cyc}a}{\sum_{cyc}ab}+2(\lambda+1)abc\left(\sum_{cyc}a\right).\frac{\sum_{cyc}a}{\sum_{cyc}ab}$$

$$\overset{?}{\ge}9\lambda abc\ (\because a+b+c=ab+bc+ca)$$

$$\Leftrightarrow(\lambda-2)\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)+2\lambda abc\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)+2abc\left(\sum_{cyc}a\right)^2$$

$$\overset{?}{\ge}3\lambda abc\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow\lambda\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2\right)-3abc\left(\sum_{cyc}ab\right)\right)+2\lambda abc\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)$$

$$\underset{(**)}{\overset{?}{\ge}}2\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2b^2-abc\left(\sum_{cyc}a\right)\right)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \text{ and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \to (4)$$

and via $(1),(2),(3)$ and (4), and $\because \lambda \geq 2 \therefore$ LHS of $(**)$ − RHS of $(**) \geq$

$$2\left(sr^2((4R+r)^2 - 2s^2) - 3r^2 s(4Rr + r^2)\right) + 4r^2 s\left(s^2 - 3(4Rr + r^2)\right)$$

$$-2s(r^2((4R+r)^2 - 2s^2) - r^2 s^2) \overset{?}{\geq} 0 \overset{\text{expanding and simplifying}}{\Leftrightarrow}$$

$$6r^2 s(s^2 - 12Rr - 3r^2) \overset{?}{\geq} 0 \to \text{true} \because s^2 - 12Rr - 3r^2$$

$$= s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq} 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{a}{\lambda a + bc} \leq \frac{3}{\lambda + 1} \; \forall a, b, c > 0 \,\middle|\, a + b + c = ab + bc + ca \text{ and } \lambda \geq 2,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$