ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ with $ab = 1$ and $\lambda \geq 0$, then :

$$\frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}\geq 2$$

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$$\frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{a(a^2+\lambda b^2-\lambda b^2)+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b(b^2+\lambda a^2-\lambda a^2)+\lambda a^3}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{1}{a}+\frac{\lambda b^3-\lambda ab^2}{a^2(a^2+\lambda b^2)}+\frac{1}{b}+\frac{\lambda a^3-\lambda a^2 b}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda\left(\frac{a^2(a-b)}{b^2(b^2+\lambda a^2)}-\frac{b^2(a-b)}{a^2(a^2+\lambda b^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{a^4(a^2+\lambda b^2)-b^4(b^2+\lambda a^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{a^6-b^6+\lambda a^2b^2(a^2-b^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{(a^2-b^2)(a^4+b^4+a^2b^2+\lambda a^2b^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a+b)(a-b)^2\left(\frac{a^4+b^4+a^2b^2(1+\lambda)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)\geq\frac{1}{a}+\frac{1}{b}$$

$$(\because a,b>0 \text{ and } \lambda\geq 0)\overset{A-G}{\geq}2.\sqrt{\frac{1}{ab}}\overset{ab=1}{=}2\ \therefore\ \frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}\geq 2$$

$$\forall\, a,b>0\,\middle|\,ab=1 \text{ and } \lambda\geq 0,''=''\ \text{iff } a=b=c=1\ (\text{QED})$$