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If $a,b,c>0$ with $a^3+b^3+c^3=3$ and $\lambda,n\in\mathbb{N}$ with $\lambda\le 3n$, then :

$$\lambda(a+b+c)+n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge 3(\lambda+n)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda(a+b+c)+n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\overset{\text{Radon}}{\ge}\lambda\left(\sum_{cyc}a\right)+n.\frac{81}{\left(\sum_{cyc}a\right)^3}\overset{a^3+b^3+c^3=3}{=}$$

$$\lambda\left(\sum_{cyc}a\right)+9n.\frac{\sum_{cyc}a^3}{\left(\sum_{cyc}a\right)^3}.\sqrt[3]{9\sum_{cyc}a^3}\overset{?}{\ge}3(\lambda+n)\overset{a^3+b^3+c^3=3}{=}3(\lambda+n).\sqrt[3]{\frac{\sum_{cyc}a^3}{3}}$$

$$\Leftrightarrow 3n\left(\frac{3\sum_{cyc}a^3}{\left(\sum_{cyc}a\right)^3}.\sqrt[3]{9\sum_{cyc}a^3}-\frac{1}{3}.\sqrt[3]{9\sum_{cyc}a^3}\right)\overset{?}{\ge}\lambda\left(3.\sqrt[3]{\frac{\sum_{cyc}a^3}{3}}-\sum_{cyc}a\right)$$

$$\Leftrightarrow 3n.\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right).\sqrt[3]{9\sum_{cyc}a^3}\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 3\lambda\left(\sum_{cyc}a\right)^3.\left(\sqrt[3]{9\sum_{cyc}a^3}-\sum_{cyc}a\right)$$

$$\text{Now, }\because 9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\overset{\text{Holder}}{\ge}0\text{ and }3n\ge\lambda$$

$$\therefore 3n.\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right).\sqrt[3]{9\sum_{cyc}a^3}\ge\lambda.\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right).\sqrt[3]{9\sum_{cyc}a^3}$$

$$\overset{?}{\ge}3\lambda\left(\sum_{cyc}a\right)^3.\left(\sqrt[3]{9\sum_{cyc}a^3}-\sum_{cyc}a\right)$$

$$\Leftrightarrow\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right).\sqrt[3]{9\sum_{cyc}a^3}\ \boxed{\overset{?}{\ge}}\ 3\left(\sum_{cyc}a\right)^3.\left(\sqrt[3]{9\sum_{cyc}a^3}-\sum_{cyc}a\right)$$

$$(\because\lambda\ge 0\text{ as }\lambda,n\in\mathbb{N})$$

$$=3\left(\sum_{cyc}a\right)^3.\frac{9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3}{\sqrt[3]{81\left(\sum_{cyc}a^3\right)^2}+\left(\sum_{cyc}a\right)^2+\left(\sum_{cyc}a\right).\sqrt[3]{9\sum_{cyc}a^3}}$$

$$\Leftrightarrow \sqrt[3]{9\sum_{cyc} a^3}\cdot\left(\sqrt[3]{81\left(\sum_{cyc} a^3\right)^2}+\left(\sum_{cyc} a\right)^2+\left(\sum_{cyc} a\right)\cdot\sqrt[3]{9\sum_{cyc} a^3}\right)$$

$$\overset{?}{\geq} 3\left(\sum_{cyc} a\right)^3 \left(\because 9\sum_{cyc} a^3-\left(\sum_{cyc} a\right)^3 \geq 0\right)$$

$$\Leftrightarrow 9\sum_{cyc} a^3+\sqrt[3]{9\sum_{cyc} a^3}\cdot\left(\sum_{cyc} a\right)^2+\sqrt[3]{81\left(\sum_{cyc} a^3\right)^2}\cdot\left(\sum_{cyc} a\right) \boxed{\underset{(**)}{\overset{?}{\geq}}} 3\left(\sum_{cyc} a\right)^3$$

$$\because 9\sum_{cyc} a^3 \geq \left(\sum_{cyc} a\right)^3 \therefore \text{LHS of } (**) \geq$$

$$\left(\sum_{cyc} a\right)^3+\left(\sum_{cyc} a\right)\cdot\left(\sum_{cyc} a\right)^2+\left(\sum_{cyc} a\right)^2\cdot\left(\sum_{cyc} a\right)=3\left(\sum_{cyc} a\right)^3$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \lambda(a+b+c)+n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) \geq 3(\lambda+n)$$

$$\forall\, a,b,c>0 \,\middle|\, a^3+b^3+c^3=3 \text{ with } \lambda,n \in \mathbb{N} \,\middle|\, \lambda \leq 3n, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality can be rewritten as

$$n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}-3\right) \geq \lambda[3-(a+b+c)] \quad (1)$$

Since $a+b+c \leq \sqrt[3]{3^2(a^3+b^3+c^3)}=3$ **and** $\lambda \leq 3n$, **then**

$$RHS_{(1)} \leq 3n[3-(a+b+c)] \overset{?}{\leq} LHS_{(1)} \Leftrightarrow n\left(3(a+b+c)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}-12\right) \geq 0,$$

which is true by AM − GM inequality,

$$3(a+b+c)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} \geq 12\sqrt[12]{a^3.b^3.c^3.\frac{1}{a^3}\cdot\frac{1}{b^3}\cdot\frac{1}{c^3}}=12.$$

Equality holds iff $a=b=c=1$.

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