ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *with* $a + b + c = 1$ *and* $\lambda > 0$, *then* :

$$\sqrt{\lambda a + 1} + \sqrt{\lambda b + 1} + \sqrt{\lambda c + 1} \geq 2 + \sqrt{\lambda + 1}$$

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$$\sqrt{\lambda a + 1} + \sqrt{\lambda b + 1} + \sqrt{\lambda c + 1} \geq 2 + \sqrt{\lambda + 1}$$

$$\Leftrightarrow \sum_{cyc}(\lambda a + 1) + 2\sum_{cyc}\sqrt{(\lambda a + 1)(\lambda b + 1)} \geq 4 + \lambda + 1 + 4.\sqrt{\lambda + 1}$$

$$\overset{a+b+c=1}{\Leftrightarrow} \lambda + 3 + 2\sum_{cyc}\sqrt{(\lambda a + 1)(\lambda b + 1)} \geq 5 + \lambda + 4.\sqrt{\lambda + 1}$$

$$\Leftrightarrow \sum_{cyc}\sqrt{(\lambda a + 1)(\lambda b + 1)} \overset{(*)}{\geq} 1 + 2$$

$\sqrt{\lambda + 1}$Now, $\sum_{cyc}\sqrt{(\lambda a + 1)(\lambda b + 1)} \overset{a+b+c=1}{=} \sum_{cyc}\sqrt{(\lambda a + a + b + c)(\lambda b + a + b + c)}$

$$= \sum_{cyc}\sqrt{\left((\lambda + 1)a + b + c\right)\left(a + (\lambda + 1)b + c\right)} \overset{\text{Reverse CBS}}{\geq} \sum_{cyc}\sqrt{\left(\sqrt{\lambda + 1}.a + \sqrt{\lambda + 1}.b + c\right)^2}$$

$$= \sqrt{\lambda + 1}.\left(\sum_{cyc} a\right) + \sqrt{\lambda + 1}.\left(\sum_{cyc} a\right) + \sum_{cyc} a \overset{a+b+c=1}{=} 2\sqrt{\lambda + 1} + 1 \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{\lambda a + 1} + \sqrt{\lambda b + 1} + \sqrt{\lambda c + 1} \geq 2 + \sqrt{\lambda + 1}\ \forall\, a, b, c > 0 \,\middle|\, a + b + c = 1$$

$$\text{and } \lambda > 0, '' ='' \text{ iff } (a = b = 0, c = 1) \text{ or } (b = c = 0, a = 1)$$

$$\text{or } (c = a = 0, b = 1)\ (\text{QED})$$