ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ *such that* : $x + y + z = xyz$ *and* $\frac{3}{2} \le \lambda \le 2$, then :

$$x^2 + y^2 + z^2 + 9(\lambda - 1) \ge \lambda(xy + yz + zx)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(i)}{=} s$$

$$\Rightarrow x = s - a, y = s - b, z = s - c \Rightarrow xyz \overset{(ii)}{=} r^2 s$$

$$\text{Via such substitutions, } \sum_{cyc} xy = \sum_{cyc} (s - a)(s - b) = 4Rr + r^2$$

$$\Rightarrow \sum_{cyc} xy \overset{(1)}{=} 4Rr + r^2 \Rightarrow \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy = s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(2)}{=} s^2 - 8Rr - 2r^2$$

$$\Leftrightarrow x^2 + y^2 + z^2 + 9(\lambda - 1) \ge \lambda(xy + yz + zx)$$

$$\Leftrightarrow \sum_{cyc} x^2 . \frac{\sum_{cyc} x}{xyz} - 9 \ge \lambda\left(\sum_{cyc} xy . \frac{\sum_{cyc} x}{xyz} - 9\right) \left(\because 1 = \frac{x + y + z}{xyz}\right)$$

$$\Leftrightarrow \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - 9xyz \overset{(*)}{\ge} \lambda\left(\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right) - 9xyz\right)$$

$$\text{Now, } \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - 9xyz - \lambda\left(\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right) - 9xyz\right)$$

$$\ge s(s^2 - 8Rr - 2r^2) - 9r^2 s - 2(s(4Rr + r^2) - 9r^2 s) \left(\begin{matrix}\because 0 < \lambda \le 2 \text{ and} \\ \text{via (i), (ii), (1) and (2)}\end{matrix}\right)$$

$$= s(s^2 - 16Rr + 5r^2) \overset{\text{Gerretsen}}{\ge} 0 \Rightarrow (*) \text{ is true} \therefore x^2 + y^2 + z^2 + 9(\lambda - 1)$$

$$\ge \lambda(xy + yz + zx) \ \forall\, x, y, z > 0 \,\middle|\, x + y + z = xyz \text{ and } \frac{3}{2} \le \lambda \le 2,$$

$$'' ='' \text{ iff } x = y = z = \sqrt{3} \text{ (QED)}$$