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If $a,b,c>0$ and $\prod_{cyc}(2a+b+c)=64$ and $n\in\mathbb{N}$ then:

$$\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}}\le 3$$

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Since $a^{2n}-1$ and $a-1$ have the same sign, then

$$(a^{2n}-1)(a-1)\ge 0 \text{ or } a^{2n+1}-a+1\ge a^{2n}$$

$$\Rightarrow\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}}\le\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n}+b^{2n}}}\overset{AM-GM}{\le}\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{2a^nb^n}}=\sum_{cyc}ab.$$

Now, we have

$$64=\prod_{cyc}(2a+b+c)\overset{AM-GM}{\ge}\prod_{cyc}2\sqrt{(a+b)(a+c)}$$

$$=8\prod_{cyc}(b+c)\overset{AM-GM}{\ge}8.\frac{8}{9}\sum_{cyc}a.\sum_{cyc}bc\ge$$

$$\ge\frac{64}{9}.\sqrt{3(ab+bc+ca)^3}\ \Rightarrow\ ab+bc+ca\le 3.$$

Therefore

$$\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}}\le 3.$$

Equality holds iff $a=b=c=1$.