ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0$ *with* $a\mathrm{b}+\mathrm{bc}+\mathrm{c}a=3$ *and* $\lambda\geq 2$, then :

$$\sum_{\text{cyc}}\frac{a^2}{a^2+\lambda\mathrm{b}}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}\frac{a^2}{a^2+\lambda\mathrm{b}}=\sum_{\text{cyc}}\frac{\left(\sqrt{a^3}\right)^2}{a^3+\lambda a\mathrm{b}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}\sqrt{a^3}\right)^2}{\sum_{\text{cyc}}a^3+\lambda\sum_{\text{cyc}}a\mathrm{b}}\overset{a\mathrm{b}+\mathrm{bc}+\mathrm{c}a=3}{=}$$

$$\frac{\sum_{\text{cyc}}a^3+2\sum_{\text{cyc}}\sqrt{a^3\mathrm{b}^3}}{\sum_{\text{cyc}}a^3+3\lambda}\geq\frac{\sum_{\text{cyc}}a^3+6\left(\frac{\sum_{\text{cyc}}a\mathrm{b}}{3}\right)^{\frac{2}{3}}}{\sum_{\text{cyc}}a^3+3\lambda}$$

$$\left(\because\left(\frac{\sum_{\text{cyc}}(a\mathrm{b})^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}}\overset{\text{Power Mean Inequality}}{\geq}\frac{\sum_{\text{cyc}}a\mathrm{b}}{3}\Rightarrow\sum_{\text{cyc}}\sqrt{a^3\mathrm{b}^3}\geq 3\left(\frac{\sum_{\text{cyc}}a\mathrm{b}}{3}\right)^{\frac{2}{3}}\right)$$

$$\overset{a\mathrm{b}+\mathrm{bc}+\mathrm{c}a=3}{=}\frac{\sum_{\text{cyc}}a^3+6}{\sum_{\text{cyc}}a^3+3\lambda}\overset{?}{\geq}\frac{3}{\lambda+1}\Leftrightarrow(\lambda-2)\left(\sum_{\text{cyc}}a^3-3\right)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\because\sum_{\text{cyc}}a^3\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a\right)\geq\frac{1}{3}\left(\sum_{\text{cyc}}a\mathrm{b}\right)\left(\sqrt{3\sum_{\text{cyc}}a\mathrm{b}}\right)\overset{a\mathrm{b}+\mathrm{bc}+\mathrm{c}a=3}{=}3$$

$$\Rightarrow\sum_{\text{cyc}}a^3-3\geq 0\ \textit{and}\ \lambda-2\geq 0\ \therefore\sum_{\text{cyc}}\frac{a^2}{a^2+\lambda\mathrm{b}}\geq\frac{3}{\lambda+1}$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0\,\middle|\, a\mathrm{b}+\mathrm{bc}+\mathrm{c}a=3\ \textit{and}\ \lambda\geq 2,''=''\ \text{iff}\ a=\mathrm{b}=\mathrm{c}=1\ (\text{QED})$$