ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ with $a^2 + b^2 + c^2 + abc = 4$, then :

$$a + b + c + ab + bc + ca \leq 6$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-India

$$\text{Now}, a^2 + b^2 + c^2 + abc = 4 \Rightarrow 4 - a^2 \overset{A-G}{\geq} 2bc + abc = bc(2 + a)$$
$$\Rightarrow 2 - a \geq bc > 0 \Rightarrow (2 - a - bc)(2 - a) \geq 0 \Rightarrow 4 - 4a + a^2 - 2bc + abc \geq 0$$
$$\Rightarrow 4a + 2bc \leq 4 + a^2 + abc \text{ and analogs} \Rightarrow 4\sum_{cyc} a + 2\sum_{cyc} ab \leq$$
$$12 + \sum_{cyc} a^2 + 3abc \Rightarrow 4\sum_{cyc} a + 4\sum_{cyc} ab \leq 12 + \sum_{cyc} a^2 + 2\sum_{cyc} ab + 3abc$$
$$\leq 12 + 3\sum_{cyc} a^2 + 3abc \overset{a^2+b^2+c^2+abc\,=\,4}{=} 12 + 3(4 - abc) + 3abc = 24$$
$$\Rightarrow a + b + c + ab + bc + ca \leq 6\ \forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 + abc = 4,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Tapas Das-India

$$a^2 + b^2 + c^2 + abc = 4 \text{ or } 4 = a^2 + b^2 + c^2 + abc \overset{AM-GM}{\geq} 4\sqrt[4]{a^3b^3c^3} \text{ or}$$
$$\sqrt[4]{a^3b^3c^3} \leq 1 \text{ or}, abc \leq 1\ (1)$$
$$a^2 + b^2 + c^2 + abc = 4 \text{ can be written as}$$
$$\frac{a}{2a + bc} + \frac{b}{2b + ac} + \frac{c}{2c + ab} = 1$$
$$\text{Now } 1 = \frac{a}{2a + bc} + \frac{b}{2b + ac} + \frac{c}{2c + ab}$$
$$= \frac{a^2}{2a^2 + abc} + \frac{b^2}{2b^2 + abc} + \frac{c^2}{2c^2 + abc} \overset{Bergstrom}{\geq}$$
$$\geq \frac{(a + b + c)^2}{2(a^2 + b^2 + c^2) + 3abc} \overset{a^2+b^2+c^2+abc=4}{=} \frac{(a + b + c)^2}{2(4 - abc) + 3abc} =$$
$$= \frac{(a + b + c)^2}{8 + abc} \overset{(1)}{\geq} \frac{(a + b + c)^2}{8 + 1} = \frac{(a + b + c)^2}{9}$$
$$\text{or } 1 \geq \frac{(a + b + c)^2}{9} \text{ or } 9 \geq (a + b + c)^2$$
$$\text{or } a + b + c \leq 3\ (2)$$
$$a + b + c + ab + bc + ca \overset{\forall x,y,z>0\ (\sum x)^2 \geq 3\sum xy}{\leq} a + b + c + \frac{(a + b + c)^2}{3} \overset{2}{\leq} 3 + \frac{3^2}{3} = 6$$
$$\text{Equality holds for } a = b = c = 1$$