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If $a,b,c>0, ab+bc+ca=3$ and $\lambda\ge 1$, then:

$$\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}\ge 3(\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}=\sum_{cyc}\sqrt{[(\lambda-1)a+(a+b)][(\lambda-1)a+(a+c)]}\ge$$

$$\ge\sum_{cyc}\left((\lambda-1)a+\sqrt{(a+b)(a+c)}\right)=(\lambda-1)\sum_{cyc}a+\sum_{cyc}\sqrt{a^2+3}\ge$$

$$\ge(\lambda-1)\sum_{cyc}a+\sum_{cyc}\frac{a+3}{\sqrt{1+3}}=\left(\lambda-\frac{1}{2}\right)\sum_{cyc}a+\frac{9}{2}\ge\left(\lambda-\frac{1}{2}\right)\sqrt{3\sum_{cyc}bc}+\frac{9}{2}=3(\lambda+1).$$

Equality holds iff $a=b=c=1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is successively equivalent to

$$2\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}\ge 2(\lambda+1)\sqrt{3(ab+bc+ca)}$$

$$2(\lambda+1)\left(a+b+c-\sqrt{3(ab+bc+ca)}\right)\ge\sum_{cyc}\left[(\lambda a+b)+(\lambda a+c)-2\sqrt{(\lambda a+b)(\lambda a+c)}\right]$$

$$\frac{2(\lambda+1)(a^2+b^2+c^2-ab-bc-ca)}{a+b+c+\sqrt{3(ab+bc+ca)}}\ge\sum_{cyc}\left(\sqrt{\lambda a+b}-\sqrt{\lambda a+c}\right)^2$$

$$\frac{(\lambda+1)[(a-b)^2+(b-c)^2+(c-a)^2]}{a+b+c+\sqrt{3(ab+bc+ca)}}\ge\sum_{cyc}\left(\frac{b-c}{\sqrt{\lambda a+b}+\sqrt{\lambda a+c}}\right)^2,$$

$$\sum_{cyc}\left(\frac{\lambda+1}{a+b+c+\sqrt{3(ab+bc+ca)}}-\frac{1}{\left(\sqrt{\lambda a+b}+\sqrt{\lambda a+c}\right)^2}\right)(b-c)^2\ge 0,$$

which is true because

$$(\lambda+1)\left(\sqrt{\lambda a+b}+\sqrt{\lambda a+c}\right)^2 \ge 2\left(\sqrt{a+b}+\sqrt{a+c}\right)^2 \ge 2(a+b+c)$$

$$\ge a+b+c+\sqrt{3(ab+bc+ca)} \text{ (and analogs)}$$

Equality holds iff $a=b=c=1$.