If $a, b, c > 0$ *such that* : $a + b + c = 1$ *and* $1 \le \lambda \le 3$, then :

$$\sum_{cyc} \frac{1}{\lambda a^2 + b + c} \le \frac{27}{\lambda + 6}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, x \in (0,1) \text{ and } \forall\, \lambda \in [1,3], \frac{1}{\lambda(1-x)^2 + x} \overset{?}{\le} \frac{9}{\lambda+6} - \frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2}$$

$$= \frac{9(\lambda+6) - 9(3-2\lambda)(3x-2)}{(\lambda+6)^2}$$

$$\Leftrightarrow \lambda^2(54x^3 - 135x^2 + 108x - 28) - \lambda(81x^3 - 324x^2 + 324x - 96)$$

$$-(81x^2 - 108x + 36) \overset{?}{\ge} 0$$

$$\Leftrightarrow \lambda^2(6x-7)(3x-2)^2 - 3\lambda(3x-8)(3x-2)^2 - 9(3x-2)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \lambda^2(6x-7) - 3\lambda(3x-8) - 9 \underset{(*)}{\underbrace{\overset{?}{\ge}}} 0$$

$\boxed{\text{Case 1}}$ $2\lambda < 3$ and we have : $\lambda^2(6x-7) - 3\lambda(3x-8) - 9$

$$= x(9\lambda^2 - 9\lambda) - 3x\lambda^2 - 7\lambda^2 + 24\lambda - 9 \overset{\lambda \ge 1 \text{ and } x<1}{\ge}$$

$$-10\lambda^2 + 24\lambda - 9 = (3-2\lambda)(5(\lambda-1)+2) + 3\lambda$$

$$\overset{1 \le \lambda < \frac{3}{2}}{\ge} 3\lambda > 0 \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $2\lambda \ge 3$ and we have : $\lambda^2(6x-7) - 3\lambda(3x-8) - 9$

$$= \lambda(3x(2\lambda-3) + 24 - 7\lambda) - 9 \overset{2\lambda \ge 3}{\ge} 24\lambda - 7\lambda^2 - 9 = (3-\lambda)(7(\lambda-1)+4) \ge 0$$

$$\left(\because 1 < \frac{3}{2} \le \lambda \le 3\right) \Rightarrow (*) \text{ is true } \therefore \text{ combining cases 1 and 2, } (*) \Rightarrow$$

$$\frac{1}{\lambda(1-x)^2 + x} \le \frac{9}{\lambda+6} - \frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2} \text{ is true } \forall\, x \in (0,1) \text{ and } \forall\, \lambda \in [1,3]$$

$$\to (1) \therefore \sum_{cyc} \frac{1}{\lambda a^2 + b + c} \overset{a+b+c=1}{=} \sum_{cyc} \frac{1}{\lambda\big(1-(b+c)\big)^2 + b + c}$$

$$= \sum_{cyc} \frac{1}{\lambda(1-x)^2 + x} \; (x = b+c, y = c+a, z = a+b \text{ and } x, y, z \in (0,1))$$

$$\overset{\text{via (1) and analogs}}{\le} \sum_{cyc} \left(\frac{9}{\lambda+6} - \frac{9(3-2\lambda)(3x-2)}{(\lambda+6)^2}\right)$$

$$= \frac{27}{\lambda+6} - \frac{9(3-2\lambda)}{(\lambda+6)^2} * \left(3\sum_{cyc} x - 6\right) = \frac{27}{\lambda+6} - \frac{9(3-2\lambda)}{(\lambda+6)^2} * \left(3\sum_{cyc}(b+c) - 6\right)$$

$$\stackrel{a+b+c=1}{=} \frac{27}{\lambda+6} - \frac{9(3-2\lambda)}{(\lambda+6)^2} * (6-6) = \frac{27}{\lambda+6}$$

$$\therefore \sum_{cyc} \frac{1}{\lambda a^2 + b + c} \le \frac{27}{\lambda+6}\ \forall\, a,b,c > 0\ \big|\, a+b+c = 1 \text{ and } 1 \le \lambda \le 3,$$

$$'' ='' \text{ iff } x = y = z = \frac{2}{3} \Rightarrow \text{iff } a = b = c = \frac{1}{3}\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove a lemma that for all $a \in (0,1)$ and $1 \le \lambda \le 3$,

$$\frac{1}{\lambda a^2 + 1 - a} \le \frac{27[\lambda + 1 - (2\lambda - 3)a]}{(\lambda+6)^2} \qquad (1)$$

We have

$$27[\lambda+1-(2\lambda-3)a](\lambda a^2+1-a)-(\lambda+6)^2$$
$$= -\lambda^2 + 15\lambda - 9 - 27(3\lambda-2)a + 27(\lambda^2+3\lambda-3)a^2 - 27\lambda(2\lambda-3)a^3$$
$$= (3a-1)^2[-\lambda^2+15\lambda-9-3\lambda(2\lambda-3)a] \ge 0,$$

because $-\lambda^2 + 15\lambda - 9 - 3\lambda(2\lambda-3)a$
$$= [(9-\lambda)(\lambda-1)+5\lambda](1-a) + (3-\lambda)(7\lambda-3)a \ge 0.$$

completing the proof of (1). Equality holds iff $a = 1$.

Returning to the proposed inequality, by using (1), we have

$$\sum_{cyc} \frac{1}{\lambda a^2 + b + c} = \sum_{cyc} \frac{1}{\lambda a^2 + 1 - a} \le \sum_{cyc} \frac{27[\lambda+1-(2\lambda-3)a]}{(\lambda+6)^2} = \frac{27}{\lambda+6}.$$

Equality holds iff $a = b = c = 1$.