ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c,d\in\left[\frac{1}{2},1\right]$ and $a\ge b\ge c\ge d$. Prove that:

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\ge\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}+4$$

Proposed by Minh Vu-Vietnam

Solution by Nguyen Van Canh-Vietnam

Let $f(a,b,c,d)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)-\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right)-4;$

$$f'_a=\frac{2}{b}-\frac{1}{d}-\frac{2d}{a^2}+\frac{b}{a^2}=\frac{2d-b}{bc}-\frac{2d-b}{a^2}=(2d-b)\left(\frac{a^2-bc}{a^2bc}\right)\ge 0$$

(since: $b,d\in\left[\frac{1}{2},1\right]\Rightarrow 2d-b\ge 1-b\ge 0$ and $a\ge b\ge c\Rightarrow a^2-bc\ge 0$)

$$\Rightarrow f(a,b,c,d)\ge f(b,b,c,d)=2\left(\frac{b}{c}+\frac{c}{d}+\frac{d}{b}\right)-\left(\frac{c}{b}+\frac{d}{c}+\frac{b}{d}\right)-3=g(b,c,d).$$

$$g'_b=\frac{2}{c}-\frac{2d}{b^2}+\frac{c}{b^2}-\frac{1}{d}=\frac{2d-c}{cd}-\frac{2d-c}{b^2}=(2d-c)\left(\frac{b^2-cd}{b^2cd}\right)\ge 0$$

(since: $c,d\in\left[\frac{1}{2},1\right]\Rightarrow 2d-c\ge 1-c\ge 0$ and $b\ge c\ge d\Rightarrow b^2-cd\ge 0$)

$$\Rightarrow g(b,c,d)\ge g(c,c,d)=2\left(\frac{c}{d}+\frac{d}{c}\right)-\left(\frac{d}{c}+\frac{c}{d}\right)-2=\frac{d}{c}+\frac{c}{d}-2\overset{\text{AM-GM}}{\ge}2-2=0.$$

$$\Rightarrow f(a,b,c,d)\ge g(b,c,d)\ge 0.$$

Proved. Equality $\Leftrightarrow a=b=c=d\in\left[\frac{1}{2};1\right]$.