ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 1$ then:

$$\log_a(bc) + \log_b(ca) + \log_c(ab) \geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be:

$$x = \ln a, y = \ln b, z = \ln c$$

$$a, b, c > 1 \Rightarrow x, y, z > 0$$

$$\log_a(bc) + \log_b(ca) + \log_c(ab) = \frac{\ln(bc)}{\ln a} + \frac{\ln(ca)}{\ln b} + \frac{\ln(ab)}{\ln c} =$$

$$= \frac{\ln b + \ln c}{\ln a} + \frac{\ln c + \ln a}{\ln b} + \frac{\ln a + \ln b}{\ln c} = \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} =$$

$$= \left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right) \overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\frac{x}{y} \cdot \frac{y}{x}} + 2\sqrt{\frac{y}{z} \cdot \frac{z}{y}} + 2\sqrt{\frac{z}{x} \cdot \frac{x}{z}} = 2 + 2 + 2 = 6$$

Equality holds for $a = b = c$.