ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ then:

$$\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2}$$

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WLOG: $a\geq b$. Denote $x=\frac{a}{b}\geq 1$.

$$\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}\geq 2+\sqrt{2},\qquad \frac{a}{b}+\frac{b}{a}+\frac{\frac{a}{b}+1}{\sqrt{\left(\frac{a}{b}\right)^2+1}}\geq 2+\sqrt{2}$$

$$x+\frac{1}{x}+\frac{x+1}{\sqrt{x^2+1}}\geq 2+\sqrt{2}$$

Let be:

$$f(x)=x+\frac{1}{x}+\frac{x+1}{\sqrt{x^2+1}},\qquad f'(x)=1-\frac{1}{x^2}+\frac{\sqrt{x^2+1}-\frac{(x+1)x}{\sqrt{x^2+1}}}{x^2+1}$$

$$f'(x)=\frac{(x-1)(x+1)}{x^2}+\frac{x^2+1-(x+1)x}{(x^2+1)\sqrt{x^2+1}}$$

$$f'(x)=(x-1)\left(\frac{x+1}{x^2}-\frac{1}{(x^2+1)\sqrt{x^2+1}}\right)$$

We will prove that:

$$\frac{x+1}{x^2}-\frac{1}{(x^2+1)\sqrt{x^2+1}}>0,\qquad x>0$$

$$\frac{x+1}{x^2}>\frac{1}{(x^2+1)\sqrt{x^2+1}},\qquad (x+1)(x^2+1)\sqrt{x^2+1}>x^2$$

$$(x+1)^2(x^2+1)^3>x^4$$

$$(x+1)^2(x^2+1)^3\overset{AM-GM}{\geq}\left(2\sqrt{x}\right)^2(2x)^3=32x^4>x^4$$

$$f'(x)\geq 0\Rightarrow f\text{ -increasing,}$$

$$\underbrace{min f(x)}_{x\geq 1}=f(1)=2+\sqrt{2}\Rightarrow f(x)\geq 2+\sqrt{2}$$

Equality holds for $x=1\Leftrightarrow a=b$.