ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c>0: a^2+b^2+c^2=a+b+c$. Prove that:

$$\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge\frac{3\sqrt{3}}{2}$$

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$$\sqrt{2a^2+bc}=\sqrt{a^2+a^2+\left(\sqrt{bc}\right)^2}\ge\sqrt{\frac{\left(a+a+\sqrt{bc}\right)^2}{3}}\overset{G-H}{\ge}\frac{2a+\frac{2bc}{b+c}}{\sqrt{3}}$$

$$=\frac{2\left(\sum_{cyc}ab\right)}{\sqrt{3}(b+c)}\text{ and analogs}\Rightarrow\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge$$

$$\frac{2\left(\sum_{cyc}ab\right)}{\sqrt{3}}\cdot\sum_{cyc}\frac{1}{(b+c)(a^2+bc)}\overset{Bergstrom}{\ge}\frac{2\left(\sum_{cyc}ab\right)}{\sqrt{3}}\cdot\frac{9}{2\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)}$$

$$=\frac{9\left(\sum_{cyc}ab\right)}{\sqrt{3}\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)}\overset{?}{\ge}\frac{3\sqrt{3}}{2}\Leftrightarrow 2\sum_{cyc}ab\overset{?}{\ge}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc$$

$$\overset{\sum_{cyc}a^2=\sum_{cyc}a}{\Leftrightarrow}2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\overset{?}{\ge}\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)\left(\sum_{cyc}ab\right)$$

$$-3abc\left(\sum_{cyc}a\right)\Leftrightarrow\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\overset{?}{\ge}2\sum_{cyc}a^2b^2+abc\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow\sum_{cyc}a^3b+\sum_{cyc}ab^3\overset{?}{\ge}2\sum_{cyc}a^2b^2\Leftrightarrow\sum_{cyc}ab(a-b)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore\frac{\sqrt{2a^2+bc}}{a^2+bc}+\frac{\sqrt{2b^2+ca}}{b^2+ca}+\frac{\sqrt{2c^2+ab}}{c^2+ab}\ge\frac{3\sqrt{3}}{2}$$

$$\forall\, a,b,c>0\,\middle|\,a^2+b^2+c^2=a+b+c,\text{''}=\text{''}\text{ iff }a=b=c=1\text{ (QED)}$$