ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c > 0$ and $abc = 1$. Prove that :

$$\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2}\le\frac{3}{5}$$

Proposed by Nguyen Thai An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2}=\sum_{cyc}\frac{a}{\lambda+2bc}\left(\lambda=\sum_{cyc}a^2\right)$$

$$=\sum_{cyc}\frac{a(\lambda+2ca)(\lambda+2ab)}{(\lambda+2ab)(\lambda+2bc)(\lambda+2ca)}$$

$$=\frac{\lambda^2(\sum_{cyc}a)+2\lambda(\sum_{cyc}a)(\sum_{cyc}ab)-6\lambda abc+4\lambda}{\lambda^3+2\lambda^2(\sum_{cyc}ab)+4\lambda abc(\sum_{cyc}a)+8(abc)^2}$$

$$=\frac{\lambda(\sum_{cyc}a)(\sum_{cyc}a^2+2\sum_{cyc}ab)-2\lambda abc}{\lambda^2(\sum_{cyc}a^2+2\sum_{cyc}ab)+4\lambda abc(\sum_{cyc}a)+8(abc)^2}$$

$$=\frac{(\sum_{cyc}a^2)\left((\sum_{cyc}a)^3-2abc\right)}{(\sum_{cyc}a^2)^2(\sum_{cyc}a)^2+4abc(\sum_{cyc}a)(\sum_{cyc}a^2)+8(abc)^2}\overset{(*)}{=}\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2}$$

$$\text{Now, }\sum_{cyc}ab\overset{A-G}{\ge}3\sqrt[3]{(abc)^2}\overset{abc=1}{=}3\Rightarrow\frac{\sum_{cyc}ab}{3}\ge1\Rightarrow1\overset{(**)}{\le}\sqrt{\frac{\sum_{cyc}ab}{3}}$$

$$\therefore\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2}\overset{\text{via }(*)\text{ and }(**)}{\le}$$

$$\sqrt{\frac{\sum_{cyc}ab}{3}}\cdot\frac{(\sum_{cyc}a^2)\left((\sum_{cyc}a)^3-2abc\right)}{(\sum_{cyc}a^2)^2(\sum_{cyc}a)^2+4abc(\sum_{cyc}a)(\sum_{cyc}a^2)+8(abc)^2}\overset{?}{\le}\frac{3}{5}$$

$$\Leftrightarrow\frac{(\sum_{cyc}ab)(\sum_{cyc}a^2)^2\left((\sum_{cyc}a)^3-2abc\right)^2}{\left((\sum_{cyc}a^2)^2(\sum_{cyc}a)^2+4abc(\sum_{cyc}a)(\sum_{cyc}a^2)+8(abc)^2\right)^2}\underset{(\bullet)}{\overset{?}{\le}}\frac{27}{25}$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$

$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)$

$\Rightarrow a=s-x, b=s-y, c=s-z\therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow$ $\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3)$ and

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4) \therefore \text{via } (1),(2),(3) \text{ and } (4), (\bullet)$$

$$\Leftrightarrow \frac{(4Rr + r^2)(s^2 - 8Rr - 2r^2)^2(s^3 - 2r^2 s)^2}{(s^2(s^2 - 8Rr - 2r^2)^2 + 4r^2s^2(s^2 - 8Rr - 2r^2) + 8r^4s^2)^2} \le \frac{27}{25}$$

$$\Leftrightarrow 27s^{10} - (964Rr + 25r^2)s^8 + r^2(11968R^2 + 1200Rr + 416r^2)s^6$$
$$-r^3(61696R^3 + 11200R^2r + 8256Rr^2 + 600r^3)s^4$$
$$+r^4(110592R^4 + 25600R^3r + 39424R^2r^2 + 8000Rr^3 + 1232r^4)s^2$$
$$-r^7(25600R^3 + 19200R^2r + 4800Rr^2 + 400r^3) \underset{(\bullet\bullet)}{\overset{?}{\ge}} 0$$

$$\text{Now, via Gerretsen, } 27(s^2 - 16Rr + 5r^2)^5$$
$$+(1196Rr - 700r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+r^2(19392R^2 - 24320Rr + 7666r^2)(s^2 - 16Rr + 5r^2)^3$$
$$+r^3(137984R^3 - 282880R^2r + 197112Rr^2 - 44340r^3) \underset{(\blacksquare)}{\overset{?}{\ge}} 0$$

$$\therefore \text{ in order to prove } (\bullet\bullet), \text{ it suffices to prove : LHS of } (\bullet\bullet) \ge \text{LHS of } (\blacksquare) \Leftrightarrow$$

$$\boxed{(380928R^4 - 1200640R^3r + 1469120R^2r^2 - 756320Rr^3 + 135307r^4)s^2 \overset{(\bullet\bullet\bullet)}{\ge} r\begin{pmatrix}5963776R^5 - 20029440R^4r + 26759976R^3r^2 \\ -16704320R^2r^3 + 4785300Rr^4 - 502975r^5\end{pmatrix}}$$

$$\because 380928R^4 - 1200640R^3r + 1469120R^2r^2 - 756320Rr^3 + 135307r^4$$
$$= (R - 2r)\begin{pmatrix}161536R^3 + 219392R^2(R - 2r) + \\ 591552Rr^2 + 426784r^3\end{pmatrix} + 988875r^4 \overset{\text{Euler}}{\ge} 988875r^3 > 0$$

$$\therefore \text{LHS of } (\bullet\bullet\bullet) \overset{\text{Rouche}}{\ge} \begin{pmatrix}380928R^4 - 1200640R^3r + \\ 1469120R^2r^2 - 756320Rr^3 + 135307r^4\end{pmatrix}\begin{pmatrix}2R^2 + 10Rr \\ -r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}\end{pmatrix}$$

$$\overset{?}{\ge} \text{RHS of } (\bullet\bullet\bullet) \Leftrightarrow 761856R^6 - 4555776R^5r + 10580352R^4r^2$$
$$-12380736R^3r^3 + 7942614R^2r^4 - 2675910Rr^5 + 367668r^6$$
$$\overset{?}{\ge} 2(R - 2r)\begin{pmatrix}380928R^4 - 1200640R^3r + 1469120R^2r^2 \\ -756320Rr^3 + 135307r^4\end{pmatrix}\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow \boxed{\begin{matrix}(R - 2r)\begin{pmatrix}761856R^5 - 3032064R^4r + 4516224R^3r^2 \\ -3348288R^2r^3 + 1246038Rr^4 - 183834r^5\end{pmatrix} \overset{?}{\ge} \\ 2(R - 2r)\begin{pmatrix}380928R^4 - 1200640R^3r + 1469120R^2r^2 \\ -756320Rr^3 + 135307r^4\end{pmatrix}\sqrt{R^2 - 2Rr}\end{matrix}} \Leftrightarrow$$

$$761856R^5 - 3032064R^4r + 4516224R^3r^2 - 3348288R^2r^3 + 1246038Rr^4$$
$$-183834r^5 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\ge}} 2\begin{pmatrix}380928R^4 - 1200640R^3r + 1469120R^2r^2 \\ -756320Rr^3 + 135307r^4\end{pmatrix}\sqrt{R^2 - 2Rr}$$

$$\left(\because R - 2r \overset{\text{Euler}}{\ge} 0\right)$$

$$\because 761856R^5 - 3032064R^4r + 4516224R^3r^2$$
$$-3348288R^2r^3 + 1246038Rr^4 - 183834r^5$$

$$= (R-2r)\left((R-2r)\begin{pmatrix}761856R^3+15360R^2r+\\1530240Rr^2+2711232r^3\end{pmatrix}+5970006r^4\right)+911250r^5$$

$$\overset{Euler}{\geq} 911250r^5 > 0 \therefore (\bullet\bullet\bullet\bullet)$$

$$\Leftrightarrow \begin{pmatrix}761856R^5-3032064R^4r+4516224R^3r^2\\-3348288R^2r^3+1246038Rr^4-183834r^5\end{pmatrix}^2 \geq$$

$$4(R^2-2Rr)\begin{pmatrix}380928R^4-1200640R^3r+\\1469120R^2r^2-756320Rr^3+135307r^4\end{pmatrix}^2$$

$$\Leftrightarrow 49928994816t^9-371514671104t^8+1103367241728t^7$$
$$-1635652599808t^6+1182357585920t^5-168754001920t^4$$
$$-357325258240t^3+268267717248t^2$$
$$-77916106348t+8448734889 \geq 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow \boxed{(t-2)\left((t-2)\left((t-2).P+1887506976384\right)+764336452500\right)+207594140625 \geq 0}$$

$$\begin{pmatrix}\text{where } P=13958643712t^6+35970351104(t-2)t^5+72575090688t^4\\+62518329344t^3+111040856064t^2+327871907840t+777562550784\end{pmatrix}$$

$$\overset{Euler}{\geq} 207594140625>0 \Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\Rightarrow \frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(c+a)^2}+\frac{c}{c^2+(a+b)^2} \leq \frac{3}{5}$$

$$\forall a,b,c>0 \,\middle|\, abc=1,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Nguyen Van Canh-Vietnam

We have:

$$\frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(a+c)^2}+\frac{c}{c^2+(a+b)^2} \leq \frac{3}{5};$$

$$\Leftrightarrow \frac{a}{a^2+(b+c)^2}+\frac{b}{b^2+(a+c)^2}+\frac{c}{c^2+(a+b)^2} \leq \frac{3}{5\sqrt[3]{abc}}; (*)$$

WLOG, we suppose that: $a^2+b^2+c^2=3$**. We have:**

$$(*) \Leftrightarrow \frac{a}{3+2bc}+\frac{b}{3+2ac}+\frac{c}{3+2ab} \leq \frac{3}{5\sqrt[3]{abc}};$$

$$\Leftrightarrow \frac{12abc+6\left(ab(a+b)+bc(b+c)+ca(c+a)\right)+9(a+b+c)}{8(abc)^2+12abc(a+b+c)+18(ab+bc+ca)+27} \leq \frac{3}{5\sqrt[3]{abc}};$$

$$\Leftrightarrow \frac{12r^3+6(pq-3r)+9p}{8r^6+12pr+18q+27} \leq \frac{3}{5r};$$

$$\left(\begin{matrix}\therefore \text{ where}: p=a+b+c\leq 3; q=ab+bc+ca; 0<r=abc\leq 1;\ p^2-2q=3\\ \Rightarrow q=\dfrac{p^2-3}{2}\end{matrix}\right)$$

$$\Leftrightarrow \frac{3\left(p^3+4r^3-6r\right)}{9p^2+12pr+8r^6} \leq \frac{3}{5r};$$

$$\Leftrightarrow 5r\left(p^3+4r^3-6r\right) \leq 9p^2+12pr+8r^6;$$

$$\Leftrightarrow 5rp^3-9p^2-12rp-8r^6+20r^4-30r^2 \leq 0;$$

$$\therefore f(p)=5rp^3-9p^2-12rp-8r^6+20r^4-30r^2, (0<p\leq 3)$$

$$\Rightarrow f'(p) = 3\left(5\,r\,p^2 - 4\,r - 6\,p\right) = 0 \overset{p>0}{\Leftrightarrow} p_0 = \frac{3+\sqrt{20r^2+9}}{5r}$$

- **Case 1:** $\frac{3+\sqrt{20r^2+9}}{5r} \le 3 \Leftrightarrow \frac{18}{41} \le r \le 1$. **We have:**

p	0	$\frac{3+\sqrt{20r^2+9}}{5r}$	3
$f'(p)$	$-$	0	$+$
$f(p)$	↘		↗

- **Case 2:** $\frac{3+\sqrt{20r^2+9}}{5r} > 3 \Leftrightarrow 0 < r < \frac{18}{41}$.

p	0	3	$\frac{3+\sqrt{20r^2+9}}{5r}$
$f'(p)$	$-$	$-$	0
$f(p)$	↘		

From Case 1 & Case 2 we have: $f(p) \le \max\{f(0), f(3)\}$.

- $f(0) = -8r^6 + 20r^4 - 30r^2 = -2\,r^2\left(4\,r^4 - 10\,r^2 + 15\right) = -2\,r^2\left(4\left(r^2 - \frac{5}{4}\right)^2 + \frac{35}{4}\right) < 0$
- $f(3) = -8\,r^6 + 20\,r^4 - 30\,r^2 + 99\,r - 81$

$$= (1-r)\left(8\,r^5 + 8\,r^4 - 12\,r^3 - 12\,r^2 + 18\,r - 81\right)$$

$$= (1-r)\left[\left(r^3+r^2\right)\left(8r^2-12\right) + 18r - 81\right] \le 0, (0 < r \le 1).$$

Therefore, $f(p) \le \max\{f(0), f(3)\} \le 0 \Rightarrow (*)\ TRUE$. **Proved. Equality**$\Leftrightarrow q = p = 3, r = 1 \Leftrightarrow a = b = c = 1$.