ROMANIAN MATHEMATICAL MAGAZINE

Find all values of k such that

$$\sum_{cyc}\frac{a^2}{b^2+c^2}+\sum_{cyc}\sqrt{\frac{a}{b+c}}\geq k\sum_{cyc}\frac{a}{b+c}$$

is true for all $a,b,c>0$.

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For $b=c=1$, we have $k\leq\frac{\frac{a^2}{2}+\frac{2}{a^2+1}+\sqrt{\frac{a}{2}}+\frac{2}{\sqrt{a+1}}}{\frac{a}{2}+\frac{2}{a+1}}\underbrace{\to}_{a\to0^+}2.$

Let us prove that the given inequality is true for $k=2$.

WLOG, we assume that $ab+bc+ca=1$. We have

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}=\sum_{cyc}\frac{2a}{2\sqrt{a(b+c)}}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2a}{a+(b+c)}=2=2(ab+bc+ca)^2\quad(1)$$

Now, by AM − GM inequality, we have

$$\frac{a^2}{b^2+c^2}+a^2(b^2+c^2)\geq2a^2\ \Rightarrow\frac{a^2}{b^2+c^2}\geq2a^2-a^2(b^2+c^2)\ \text{(and analogs)}$$

$$\Rightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}\geq2(a^2+b^2+c^2)-2(a^2b^2+b^2c^2+c^2a^2)\quad(2)$$

From the results (1) and (2), it suffices to prove that

$$a^2+b^2+c^2+2abc(a+b+c)\geq\sum_{cyc}\frac{a}{b+c}.$$

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a(ab+bc+ca)}{b+c}=\sum_{cyc}a^2+abc\sum_{cyc}\frac{1}{b+c}=\sum_{cyc}a^2+abc\sum_{cyc}\left(a+\frac{bc}{b+c}\right)$$

$$\overset{HM-AM}{\leq}\sum_{cyc}a^2+abc\sum_{cyc}\left(a+\frac{b+c}{4}\right)\leq a^2+b^2+c^2+2abc(a+b+c).$$

So the given inequality is true for

$k=2$. Therefore, the given inequality is true for all $\boxed{k\leq2}$.