ROMANIAN MATHEMATICAL MAGAZINE

Find all values of k such that

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}+\sum_{cyc}\frac{a^3}{b^3+c^3}\geq k\sum_{cyc}\frac{a^2}{b^2+c^2},$$

is true for all $a,b,c>0$.

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For $b=c=1$, we have

$$k\leq\frac{\sqrt{\frac{a}{2}}+\frac{2}{\sqrt{a+1}}+\frac{a^3}{2}+\frac{2}{a^3+1}}{\frac{a^2}{2}+\frac{2}{a^2+1}}\underbrace{\to}_{a\to0^+}2.$$ **We will prove that the values of k are**

$k\leq2$. For this, it is suffices to prove that the given inequality is true for $k=2$.

We have: $$\sum_{cyc}\sqrt{\frac{a}{b+c}}=2\sum_{cyc}\frac{a}{2\sqrt{a(b+c)}}\overset{AM-GM}{\geq}2\sum_{cyc}\frac{a}{a+(b+c)}=2.$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}=1+\frac{a^6+b^6+c^6+a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}=1+\frac{a^6+b^6+c^6+a^2b^2c^2}{\sum_{cyc}a^2b^2(a^2+b^2)+2a^2b^2c^2}\leq$$

$$\leq1+\frac{a^6+b^6+c^6+a^2b^2c^2}{2(a^3b^3+b^3c^3+c^3a^3)+2a^2b^2c^2}.$$

From these results, it suffices to prove that

$$\left(a^3b^3+b^3c^3+c^3a^3+a^2b^2c^2\right)\sum_{cyc}\frac{a^3}{b^3+c^3}\geq a^6+b^6+c^6+a^2b^2c^2$$

$$\Leftrightarrow a^3b^3c^3\sum_{cyc}\frac{1}{b^3+c^3}+a^2b^2c^2\sum_{cyc}\frac{a^3}{b^3+c^3}\geq a^2b^2c^2,$$

which is true because $\sum_{cyc}\frac{a^3}{b^3+c^3}\geq\sum_{cyc}\frac{a^3}{a^3+b^3+c^3}=1.$

Therefore, the values of k such that the given inequality is true for all $a,b,c>0$ are

$$k\leq2.$$