ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c \geq 0$ such that : $ab+bc+ca=1$. Prove that :

$$\frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2) \geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

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$$a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2) \overset{1=ab+bc+ca}{=}$$
$$a\left(\sum_{cyc} ab-b^2\right)\left(\sum_{cyc} ab-c^2\right)+b\left(\sum_{cyc} ab-c^2\right)\left(\sum_{cyc} ab-a^2\right)$$
$$+c\left(\sum_{cyc} ab-a^2\right)\left(\sum_{cyc} ab-b^2\right)=4abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2) \overset{(*)}{=} 4abc\left(\sum_{cyc} ab\right)$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)$$

$$\overset{\text{via }(*)}{\Leftrightarrow} \frac{4(a^3+b^3+c^3)}{3} \geq 4abc\left(\sum_{cyc} ab\right) \overset{1=ab+bc+ca}{=} 4abc \Leftrightarrow a^3+b^3+c^3-3abc$$

$$\geq 0 \Leftrightarrow \frac{(a+b+c)}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \geq 0 \rightarrow \text{true} \because a,b,c \geq 0$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2)$$

$$\text{Also, } a(1-b^2)(1-c^2)+b(1-c^2)(1-a^2)+c(1-a^2)(1-b^2) \geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

$$\overset{\text{via }(*)}{\Leftrightarrow} 4abc\left(\sum_{cyc} ab\right)^4 \geq 108a^3b^3c^3 \Leftrightarrow \left(\sum_{cyc} ab\right)^3 \overset{(i)}{\geq} 27a^2b^2c^2$$

$$\text{Now, } (x+y+z)^3-27xyz=\sum_{cyc} x^3+3\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)-xyz\right)-27xyz$$

$$=3xyz+\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2-\sum_{cyc} xy\right)-3xyz+3\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)-27xyz$$

$$= \left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) + 3\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right)$$

$$= \left(\sum_{cyc} x\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) + 3\left(\begin{matrix}(y^2x + z^2x - 2xyz) + (z^2y + x^2y - 2xyz) + \\ (x^2z + y^2z - 2xyz)\end{matrix}\right)$$

$$= \frac{(\sum_{cyc} x)}{2}.\sum_{cyc}(x-y)^2 + 3\left(\sum_{cyc} x(y-z)^2\right) \geq 0 \ \forall\, x, y, z \geq 0 \text{ and choosing}$$

$$x = ab, y = bc, z = ca, \text{we arrive at} : \left(\sum_{cyc} ab\right)^3 \geq 27a^2b^2c^2 \Rightarrow \text{(i) is true}$$

$$\therefore a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2) \geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}$$

$$\therefore \frac{4(a^3+b^3+c^3)}{3} \geq a(1-b^2)(1-c^2) + b(1-c^2)(1-a^2) + c(1-a^2)(1-b^2)$$

$$\geq \frac{108}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3} \ \forall\, a, b, c \geq 0 \,\middle|\, ab + bc + ca = 1, '' ='' \text{ iff } a = b = c = \frac{1}{\sqrt{3}} \text{ (QED)}$$