ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0$ such that $a+b+c=3$. Prove that :

$$28\min\left\{\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right),\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\right\}$$
$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

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Firstly, we *shall* prove : $28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3\geq 27(ab+bc+ca)$

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=3)$

and then : $\text{LHS}-\text{RHS}=28*3^{\frac{2023}{2024}}-3>3^3*3^{\frac{2023}{2024}}-3>0$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=3)$

and then : LHS − RHS =

$$28.\left(\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)-3-\frac{27.9bc}{(b+c)^2}$$

$$\overset{G-H}{\geq}28\left(\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3-\frac{243bc}{(b+c)^2}$$

$$=28.2024\left(\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243bc}{(b+c)^2}$$

$$\underset{\because b+c=3}{\overset{\text{Bergstrom and}}{\geq}}\frac{28.2024(b+c)^2}{\frac{2023}{3}(b+c)^2+b^2+c^2}-3-\frac{243bc}{(b+c)^2}$$

$$=\frac{(84.2024-6069)(b^2+c^2+2bc)-9(b^2+c^2)}{2026(b^2+c^2)+2023.2bc}-\frac{243bc}{b^2+c^2+2bc}$$

$$=\frac{163938x+163947y}{2026x+2023y}-\frac{243y}{2(x+y)}\quad (x=b^2+c^2; y=2bc)$$

$$=\frac{327876x^2+163452xy-163695y^2}{2(x+y)(2026x+2023y)}$$

$$=\frac{(327876x+491328y)(x-y)+327633y^2}{2(x+y)(2026x+2023y)}>0\ \because x=b^2+c^2\geq 2bc=y$$

and $x,y>0$ as $b,c>0\ \therefore 28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)-3>27(ab+bc+ca)$

$\boxed{\text{Case 3}}$ $a,b,c>0$ and then : $\text{LHS}-\text{RHS}\overset{\because a+b+c=3}{=}$

$$28.\left(\sqrt[2024]{\underbrace{a*a*a*\ldots a}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{b*b*b*\ldots b}_{2023\text{ terms}}*1}+\sqrt[2024]{\underbrace{c*c*c*\ldots c}_{2023\text{ terms}}*1}\right)-3$$

$$-\frac{27.9(ab+bc+ca)}{(a+b+c)^2}\overset{G-H}{\geq}28\left(\frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=28.2024\left(\frac{a^2}{2023a+a^2}+\frac{b^2}{2023b+b^2}+\frac{c^2}{2023c+c^2}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq}\frac{28.2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2}-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{84.2024(u+2v)}{2023(u+2v)+3u}-3-\frac{243v}{u+2v}\ (u=a^2+b^2+c^2, v=ab+bc+ca)$$

$$=\frac{6(27323u^2+27242uv-54565v^2)}{2026u+4046v}=\frac{6(u-v)(27323u+54565v)}{2026u+4046v}\geq 0$$

$$(\because a^2+b^2+c^2\geq ab+bc+ca\Rightarrow u\geq v \text{ and } a,b,c>0\Rightarrow u,v>0)$$

$$\therefore 28\left(\sum_{cyc}a^{\frac{2023}{2024}}\right)-3\geq 27\left(\sum_{cyc}ab\right) \text{ and combining all cases,}$$

$$28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)\geq 27(ab+bc+ca)+3$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

$$\left(\because a,b,c\geq 0 \text{ such that } a+b+c=3\overset{\text{via } A-G}{\Rightarrow}0\leq abc\leq 1\Rightarrow 0\leq (abc)^{\frac{2025}{2026}}\leq 1\right)$$

$$\therefore \boxed{28\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right)\overset{(*)}{\geq}27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}}$$

Now, we *shall* prove : $28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)-3\geq 27(ab+bc+ca)$

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ *and* WLOG we may assume $b=c=0$ $(a=3)$

and then : $\text{LHS}-\text{RHS}=28*3^{\frac{2024}{2025}}-3>3^3*3^{\frac{2024}{2025}}-3>0$

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ *and* WLOG we may assume $a=0$ $(b+c=3)$

and then : $\text{LHS}-\text{RHS}=$

$$28.\left(\sqrt[2025]{\underbrace{b*b*b*\ldots b}_{2024\text{ terms}}*1}+\sqrt[2025]{\underbrace{c*c*c*\ldots c}_{2024\text{ terms}}*1}\right)-3-\frac{27.9bc}{(b+c)^2}$$

$$\overset{G-H}{\geq}28\left(\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}\right)-3-\frac{243bc}{(b+c)^2}$$

$$=28.2025\left(\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)-3-\frac{243bc}{(b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b+c=3}}{\geq}\frac{28.2025(b+c)^2}{\frac{2024}{3}(b+c)^2+b^2+c^2}-3-\frac{243bc}{(b+c)^2}$$

$$=\frac{(84.2025-6072)\left(b^2+c^2+2bc\right)-9\left(b^2+c^2\right)}{2027(b^2+c^2)+2024.2bc}-\frac{243bc}{b^2+c^2+2bc}$$

$$=\frac{164019x+164028y}{2027x+2024y}-\frac{243y}{2(x+y)}\ \left(x=b^2+c^2;y=2bc\right)$$

$$=\frac{328038x^2+163533xy-163776y^2}{2(x+y)(2027x+2024y)}$$

$$=\frac{(328038x+491571y)(x-y)+327795y^2}{2(x+y)(2027x+2024y)}>0\because x=b^2+c^2\geq 2bc=y$$

$$and\ x,y>0\ as\ b,c>0\therefore 28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)-3>27(ab+bc+ca)$$

$\boxed{Case\ 3}$ $a,b,c>0$ and then : $LHS-RHS\overset{\because a+b+c=3}{=}-\frac{27.9(ab+bc+ca)}{(a+b+c)^2}-3$

$$+28.\left(\sqrt[2025]{\underbrace{a*a*a*\dots a}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{b*b*b*\dots b}_{2024\ terms}*1}+\sqrt[2025]{\underbrace{c*c*c*\dots c}_{2024\ terms}*1}\right)$$

$$\overset{G-H}{\geq}28\left(\frac{2025}{\frac{2024}{a}+1}+\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=28.2025\left(\frac{a^2}{2024a+a^2}+\frac{b^2}{2024b+b^2}+\frac{c^2}{2024c+c^2}\right)-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{\substack{\text{Bergstrom}\\ and\\ \because a+b+c=3}}{\geq}\frac{28.2025(a+b+c)^2}{\frac{2024}{3}(a+b+c)^2+a^2+b^2+c^2}-3-\frac{243(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{84.2025(u+2v)}{2024(u+2v)+3u}-3-\frac{243v}{u+2v}\ \left(u=a^2+b^2+c^2,v=ab+bc+ca\right)$$

$$=\frac{3\left(54673u^2+54511uv-109184v^2\right)}{2027u+4048v}=\frac{3(u-v)(54673u+109184v)}{2027u+4048v}\geq 0$$

$$\left(\because a^2+b^2+c^2\geq ab+bc+ca\Rightarrow u\geq v\ and\ a,b,c>0\Rightarrow u,v>0\right)$$

$$\therefore 28\left(\sum_{cyc}a^{\frac{2024}{2025}}\right)-3\geq 27\left(\sum_{cyc}ab\right)\ \text{and combining } all \text{ cases,}$$

$$28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\geq 27(ab+bc+ca)+3$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}$$

$$\left(\because a,b,c\geq 0\text{ such that }a+b+c=3\overset{via\ A-G}{\Rightarrow}0\leq abc\leq 1\Rightarrow 0\leq (abc)^{\frac{2025}{2026}}\leq 1\right)$$

$$\therefore\boxed{28\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\overset{(**)}{\geq}27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}}$$

$$\therefore (*),(**)\Rightarrow 28\min\left\{\left(a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}\right),\left(a^{\frac{2024}{2025}}+b^{\frac{2024}{2025}}+c^{\frac{2024}{2025}}\right)\right\}$$

$$\geq 27(ab+bc+ca)+3(abc)^{\frac{2025}{2026}}\ \forall\ a,b,c\geq 0\ \middle|\ a+b+c=3,$$

$$''='' \text{ iff } a=b=c=1\ \text{(QED)}$$