ROMANIAN MATHEMATICAL MAGAZINE

Mr. Bin entered the problem as follows :"

Let $a, b, c \geq 0$ such that $a + b + c = 3$. Prove that :

$$a^{\frac{2023}{2024}} + b^{\frac{2023}{2024}} + c^{\frac{2023}{2024}} + 15 \geq \left(6 - 3\sqrt{3}\right)(ab + bc + ca) + 9\sqrt{3}''$$

in the AI chat software, but the software could not solve it.

And you ? Let's solve this puzzle !

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Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables $= 0$ and WLOG we may assume $b = c = 0$

$(a = 3)$ and then : $\text{LHS} - \text{RHS} = 3^{\frac{2023}{2024}} + 15 - 9\sqrt{3} > \sqrt{3} + 15 - 9\sqrt{3} > 0$

$\boxed{\text{Case 2}}$ Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ $(b + c = 3)$

$$\text{and then : LHS} - \text{RHS} = \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1} + 15$$

$$-9\left(\frac{6-3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b+c)^2}\right) - 9\sqrt{3} \overset{G-H}{\geq} \frac{2024}{\frac{2023}{b}+1} + \frac{2024}{\frac{2023}{c}+1} + 15$$

$$-\frac{9\left(6-3\sqrt{3}\right)}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right) - 9\sqrt{3}$$

$$= \frac{2024b^2}{2023b+b^2} + \frac{2024c^2}{2023c+c^2} + 15 - 27\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-9\sqrt{3}\left(1 - \frac{3}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)\right) \underset{\because b+c=3}{\overset{\text{Bergstrom and}}{\geq}} \frac{2024(b+c)^2}{\frac{2023}{3}(b+c)^2 + b^2 + c^2} + 15$$

$$-27\left(\frac{2bc}{b^2+c^2+2bc}\right) - 9\sqrt{3}\left(1 - \frac{3}{2}\left(\frac{2bc}{b^2+c^2+2bc}\right)\right)$$

$$= \frac{2024(x+y)}{\frac{2023}{3}(x+y)+x} + 15 - 27\left(\frac{y}{x+y}\right) - 9\sqrt{3}\left(1 - \frac{3}{2}\left(\frac{y}{x+y}\right)\right)$$

$$\left(x = b^2 + c^2; y = 2bc\right) \overset{?}{\geq} 0 \Leftrightarrow \frac{3(6077x + 6068y)(2x - y)}{(x+y)(2026x + 2023y)} \underset{(*)}{\overset{?}{\geq}} \frac{9\sqrt{3}(2x - y)}{2(x+y)}$$

If $2x = y$, then : $b^2 + c^2 = bc \Rightarrow (b+c)^2 = 3bc \Rightarrow bc = 3$ and $\therefore b + c = 3$

$\therefore b + \frac{3}{b} = 3 \Rightarrow b^2 - 3b + 3 = 0$ and $\because \Delta = 9 - 12 < 0 \Rightarrow$ no real values of

b, c exist such that $b + c = 3$ and $b^2 + c^2 = bc \therefore 2x - y \neq 0$

$$\therefore (*) \Leftrightarrow \frac{6077x + 6068y}{2026x + 2023y} \geq \frac{3\sqrt{3}}{2} \text{ and } \because \frac{3\sqrt{3}}{2} < \frac{13}{5} \therefore \text{it suffices to prove :}$$

$$\frac{6077x + 6068y}{2026x + 2023y} > \frac{13}{5} \Leftrightarrow 3(1349x + 1347y) > 0 \to \text{true} \because x, y > 0$$

$$\Rightarrow (*) \text{ is true} \therefore a^{\frac{2023}{2024}} + b^{\frac{2023}{2024}} + c^{\frac{2023}{2024}} + 15 > \left(6 - 3\sqrt{3}\right)(ab + bc + ca) + 9\sqrt{3}$$

$$\boxed{\text{Case 3}}\ a,b,c>0 \text{ and then : LHS} - \text{RHS} \overset{\because a+b+c=3}{=} \sqrt[2024]{\underbrace{a.a.a\ldots a}_{2023\text{ terms}}.1}$$

$$+\sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1}+\sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1}+15-\frac{9(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}-9\sqrt{3}$$

$$\overset{G-H}{\geq}\frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}$$

$$-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)$$

$$=\frac{2024a^2}{2023a+a^2}+\frac{2024b^2}{2023b+b^2}+\frac{2024c^2}{2023c+c^2}+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}$$

$$-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq}\frac{2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2}$$

$$+15-\frac{54(ab+bc+ca)}{(a+b+c)^2}-9\sqrt{3}\left(1-\frac{3(ab+bc+ca)}{(a+b+c)^2}\right)$$

$$=\frac{2024(u+2v)}{\frac{2023}{3}(u+2v)+u}+15-\frac{54v}{u+2v}-9\sqrt{3}\left(1-\frac{3v}{u+2v}\right)\left(u=\sum_{cyc}a^2, v=\sum_{cyc}ab\right)$$

$$\overset{?}{\geq}0=\frac{3(6077u+12136v)(u-v)}{(u+2v)(1013u+2023v)}\overset{?}{\geq}\frac{9\sqrt{3}(u-v)}{u+2v}$$

$$\Leftrightarrow 6077u+12136v\overset{?}{\geq}3\sqrt{3}(1013u+2023v)\ (\because u-v\geq 0)\text{ and }\because 3\sqrt{3}<\frac{16}{5}$$

$$\therefore \text{it suffices to prove : } 6077u+12136v>\frac{16}{5}(1013u+2023v)$$

$$\Leftrightarrow 14177u+28312v>0\rightarrow \text{true}\because u,v>0$$

$$\therefore a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+15\geq(6-3\sqrt{3})(ab+bc+ca)+9\sqrt{3}$$

$$\therefore \text{combining } all \text{ cases}, a^{\frac{2023}{2024}}+b^{\frac{2023}{2024}}+c^{\frac{2023}{2024}}+15$$

$$\geq(6-3\sqrt{3})(ab+bc+ca)+9\sqrt{3}\ \forall\, a,b,c\geq 0\,\middle|\, a+b+c=3,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$