ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0$ such that $a + b + c = 3$. Prove that :

$$\text{(i) } a^{\frac{2023}{2024}} + b^{\frac{2023}{2024}} + c^{\frac{2023}{2024}} + 2(16 - 9\sqrt{3}) \geq (6 - 3\sqrt{3})(ab + bc + ca) \text{ and}$$

$$\text{(ii) } a^{\frac{2024}{2023}} + b^{\frac{2024}{2023}} + c^{\frac{2024}{2023}} + 2(16 - 9\sqrt{3}) \geq (6 - 3\sqrt{3})(ab + bc + ca)$$

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We shall first prove :

$$\text{(i) } a^{\frac{2023}{2024}} + b^{\frac{2023}{2024}} + c^{\frac{2023}{2024}} + 2(16 - 9\sqrt{3}) \geq (6 - 3\sqrt{3})(ab + bc + ca)$$

$\boxed{\text{Case 1}}$ Exactly 2 variables $= 0$ and WLOG we may assume $b = c = 0$ $(a = 3)$ and then : $\text{LHS} - \text{RHS} = 3^{\frac{2023}{2024}} + 2(16 - 9\sqrt{3}) > 0 \therefore$ (i) is true (strict inequality)

$\boxed{\text{Case 2}}$ Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ $(b + c = 3)$ and then :

$$\text{LHS} - \text{RHS} = \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1} + 2(16 - 9\sqrt{3})$$

$$-9\left(\frac{6 - 3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b + c)^2}\right) \overset{G-H}{\geq} \frac{2024}{\frac{2023}{b} + 1} + \frac{2024}{\frac{2023}{c} + 1} + 2(16 - 9\sqrt{3})$$

$$-\frac{9(6 - 3\sqrt{3})}{2}\left(\frac{2bc}{b^2 + c^2 + 2bc}\right)$$

$$= \frac{2024b^2}{2023b + b^2} + \frac{2024c^2}{2023c + c^2} + 32 - 27\left(\frac{2bc}{b^2 + c^2 + 2bc}\right)$$

$$-18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{2bc}{b^2 + c^2 + 2bc}\right) \overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because b + c = 3}}{\geq}$$

$$\frac{2024(b + c)^2}{\frac{2023}{3}(b + c)^2 + b^2 + c^2} + 32 - 27\left(\frac{2bc}{b^2 + c^2 + 2bc}\right)$$

$$-18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{2bc}{b^2 + c^2 + 2bc}\right)$$

$$= \frac{2024(x + y)}{\frac{2023}{3}(x + y) + x} + 32 - 27\left(\frac{y}{x + y}\right) - 18\sqrt{3} + \frac{27\sqrt{3}}{2}\left(\frac{y}{x + y}\right)$$

$$(x = b^2 + c^2; y = 2bc) = \frac{70904x^2 + 87010xy + 16187y^2}{(x + y)(2026x + 2023y)} - \frac{9\sqrt{3}(4x + y)}{2(x + y)}$$

$$\overset{\frac{9\sqrt{3}}{2} < 8}{>} \frac{70904x^2 + 87010xy + 16187y^2}{(x + y)(2026x + 2023y)} - \frac{8(4x + y)}{x + y} = \frac{6072x^2 + 6066xy + 3y^2}{(x + y)(2026x + 2023y)}$$

$$> 0 \because x \geq y > 0 \therefore \text{(i) is true (strict inequality)}$$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and then : $\text{LHS} - \text{RHS} \overset{\because a + b + c = 3}{=}$

$$\sqrt[2024]{\underbrace{a.a.a\ldots a}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{b.b.b\ldots b}_{2023\text{ terms}}.1} + \sqrt[2024]{\underbrace{c.c.c\ldots c}_{2023\text{ terms}}.1}$$

$$+2\left(16-9\sqrt{3}\right)-\frac{9\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{G-H}{\geq}\frac{2024}{\frac{2023}{a}+1}+\frac{2024}{\frac{2023}{b}+1}+\frac{2024}{\frac{2023}{c}+1}+2\left(16-9\sqrt{3}\right)-\frac{9\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{2024a^2}{2023a+a^2}+\frac{2024b^2}{2023b+b^2}+\frac{2024c^2}{2023c+c^2}+2\left(16-9\sqrt{3}\right)$$

$$-\frac{9\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=3}}{\geq}$$

$$\frac{2024(a+b+c)^2}{\frac{2023}{3}(a+b+c)^2+a^2+b^2+c^2}+2\left(16-9\sqrt{3}\right)-\frac{9\left(6-3\sqrt{3}\right)(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{2024(u+2v)}{\frac{2023}{3}(u+2v)+u}+32-\frac{54v}{u+2v}-18\sqrt{3}+27\sqrt{3}\left(\frac{v}{u+2v}\right)$$

$$\left(u=\sum_{cyc}a^2,v=\sum_{cyc}ab\right)=\frac{70904u^2+174020uv+64748v^2}{(u+2v)(2026u+4046v)}-\frac{9\sqrt{3}(2u+v)}{u+2v}$$

$$\overset{9\sqrt{3}<16}{>}\frac{70904u^2+174020uv+64748v^2}{(u+2v)(2026u+4046v)}-\frac{16(2u+v)}{u+2v}$$

$$=\frac{6072u^2+12132uv+12v^2}{(u+2v)(2026u+4046v)}>0\because u\geq v>0\therefore\text{(i) is true (strict inequality)}$$

$\therefore$ combining all cases, (i) is true (strict inequality) $\forall\, a,b,c\geq 0\,\middle|\, a+b+c=3$

We shall now prove :

$$\text{(ii) } a^{\frac{2024}{2023}}+b^{\frac{2024}{2023}}+c^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)\geq\left(6-3\sqrt{3}\right)(ab+bc+ca)$$

$\boxed{Case\ 1}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=3)$

and then : $\text{LHS}-\text{RHS}=3^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)>0\therefore$ (ii) is true (strict inequality)

$\boxed{Case\ 2}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=3)$

and then : $\text{LHS}-\text{RHS}=b^{\frac{2024}{2023}}+c^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)-\left(6-3\sqrt{3}\right)(bc)$

$$\overset{\substack{\text{Power-Mean inequality}\\ \text{and}\\ \text{A-G}}}{\geq}\frac{1}{2^{\frac{2024}{2023}-1}}(b+c)^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)-\frac{\left(6-3\sqrt{3}\right)}{4}(b+c)^2$$

$$\overset{b+c=3}{=}\frac{1}{2^{\frac{1}{2023}}}.3^{\frac{2024}{2023}}+2\left(16-9\sqrt{3}\right)-\frac{9\left(6-3\sqrt{3}\right)}{4}=\frac{1}{2^{\frac{1}{2023}}}.3^{\frac{2024}{2023}}+\frac{37}{2}-\frac{45\sqrt{3}}{4}$$

$$>\frac{3}{2}+\frac{37}{2}-\frac{45\sqrt{3}}{4}>\frac{1}{2}>0\therefore\text{(ii) is true (strict inequality)}$$

$\boxed{Case\ 3}$ $a,b,c>0$ and then : $\text{LHS}-\text{RHS}\overset{\substack{\text{Power-Mean inequality}\\ \text{and}\\ \text{A-G}}}{\geq}\frac{1}{3^{\frac{2024}{2023}-1}}\left(\sum_{cyc}a\right)^{\frac{2024}{2023}}$

$$+2\left(16-9\sqrt{3}\right)-\left(2-\sqrt{3}\right)(a+b+c)^2 \overset{a+b+c=3}{=} \frac{3^{\frac{2024}{2023}}}{3^{\frac{1}{2023}}}+2\left(16-9\sqrt{3}\right)$$

$$-9\left(2-\sqrt{3}\right)=3+32-18\sqrt{3}-18+9\sqrt{3}>17-9\sqrt{3}>0\ (\because 289>243)$$

$\therefore$ (ii) is true (strict inequality) $\therefore$ combining *all* cases,

(ii) is true (strict inequality) $\forall\, a,b,c\geq 0 \mid a+b+c=3$ (QED)