ROMANIAN MATHEMATICAL MAGAZINE

Let $a,\mathrm{b},\mathrm{c} \geq 0$ such that : $a+\mathrm{b}+\mathrm{c}=3$. Prove that :

$$a)\ 9\left(a^{\mathrm{k}}+\mathrm{b}^{\mathrm{k}}+\mathrm{c}^{\mathrm{k}}\right)+a\mathrm{b}+\mathrm{bc}+\mathrm{c}a \leq 27+3a\mathrm{bc} \text{ with } \mathrm{k}=\frac{1}{2} \text{ and}$$

$$\mathrm{b})\ 9\left(a^{\mathrm{k}}+\mathrm{b}^{\mathrm{k}}+\mathrm{c}^{\mathrm{k}}\right)+a\mathrm{b}+\mathrm{bc}+\mathrm{c}a \geq 27+3a\mathrm{bc} \text{ with } \mathrm{k} \geq 1$$

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When exactly 2 variables $=0$ (WLOG $\mathrm{b}=\mathrm{c}=0$ and $a=3$), then :

$$\text{LHS of } a)=9\sqrt{3}<27=\text{RHS of } a) \text{ and LHS of b})=9.3^{\mathrm{k}} \overset{\mathrm{k}\geq 1}{\geq} 27=\text{RHS of b})$$

When exactly 1 variable $=0$ (WLOG and $a=0$ and $\mathrm{b}+\mathrm{c}=3$; $\mathrm{b},\mathrm{c}>0$), then :

$$\text{LHS of } a)=9\left(\sqrt{\mathrm{b}}+\sqrt{\mathrm{c}}\right)+\mathrm{bc} \overset{\substack{\text{CBS}\\ \text{and}\\ \text{A-G}}}{\leq} 9\sqrt{2(\mathrm{b}+\mathrm{c})}+\frac{(\mathrm{b}+\mathrm{c})^2}{4} \overset{\mathrm{b}+\mathrm{c}=3}{=} 9\sqrt{6}+\frac{9}{4}<27$$

$$=\text{RHS of } a) \text{ and LHS of b})=9\left(\left(1+(\mathrm{b}-1)\right)^{\mathrm{k}}+\left(1+(\mathrm{c}-1)\right)^{\mathrm{k}}\right)+\mathrm{bc}$$

$$\overset{\textit{via}\ \text{Bernoulli}\ \because\ k\geq 1}{\geq} 9\left(2+\mathrm{k}\left((\mathrm{b}-1)+(\mathrm{c}-1)\right)\right)+\mathrm{bc} \overset{\mathrm{b}+\mathrm{c}=3}{=} 18+9\mathrm{k}(3-2)+\mathrm{bc}$$

$$\overset{\mathrm{k}\geq 1}{\geq} 27+\mathrm{bc}>27=\text{RHS of b})$$

We now consider the case when $a,\mathrm{b},\mathrm{c}>0$ and $a) \Leftrightarrow$

$$9\sum_{\text{cyc}} x+\sum_{\text{cyc}} x^2\mathrm{y}^2 \leq 27+3x^2\mathrm{y}^2\mathrm{z}^2 \ \left(x=\sqrt{a},\mathrm{y}=\sqrt{\mathrm{b}},\mathrm{z}=\sqrt{\mathrm{c}}\right)$$

$$\overset{3=\sum_{\text{cyc}} x^2}{\Leftrightarrow} \left(\sum_{\text{cyc}} x^2\right)^3+3x^2\mathrm{y}^2\mathrm{z}^2-\frac{1}{3}\left(\sum_{\text{cyc}} x^2\mathrm{y}^2\right)\left(\sum_{\text{cyc}} x^2\right)$$

$$\geq 9\sum_{\text{cyc}} x$$

$$\overset{\because\ 3=\sum_{\text{cyc}} x^2}{\Leftrightarrow} 3\left(\sum_{\text{cyc}} x^2\right)^3+9x^2\mathrm{y}^2\mathrm{z}^2-\left(\sum_{\text{cyc}} x^2\mathrm{y}^2\right)\left(\sum_{\text{cyc}} x^2\right) \geq \sqrt{3\left(\sum_{\text{cyc}} x^2\right)^5}\sum_{\text{cyc}} x$$

$$\Leftrightarrow \boxed{\left(3\left(\sum_{\text{cyc}} x^2\right)^3+9x^2\mathrm{y}^2\mathrm{z}^2-\left(\sum_{\text{cyc}} x^2\mathrm{y}^2\right)\left(\sum_{\text{cyc}} x^2\right)\right)^2 \overset{(*)}{\geq} 3\left(\sum_{\text{cyc}} x\right)^2\left(\sum_{\text{cyc}} x^2\right)^5}$$

$$\left(\begin{array}{c} \because 3\left(\sum_{cyc} a\right)^3 + 9abc - \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right) \\ 3\sum_{cyc} a^3 + 9\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - abc\right) + 9abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \\ 3\sum_{cyc} a^3 + 9\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - abc\right) + 9abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \end{array}\right)$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y$
$\Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (1)$

$\Rightarrow x = s - X, y = s - Y, z = s - Z$ and such substitutions $\Rightarrow$

$$\sum_{cyc} xy = \sum_{cyc}(s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (2) \text{ and}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (1) and (2)}}{=}$$

$$(4Rr + r^2)^2 - 2\left(\prod_{cyc}(s - X)\right).s = (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (3) \text{ and}$$

$$\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (2)}}{=} s^2 - (4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (4)$$

$\therefore$ via (1)(3) and (4), $(*)$

$$\Leftrightarrow \boxed{\begin{array}{c}\left(3\left(s^2 - 8Rr - 2r^2\right)^3 + 9r^4s^2 - r^2\left((4R + r)^2 - 2s^2\right)\left(s^2 - 8Rr - 2r^2\right)\right)^2 \\ -3s^2\left(s^2 - 8Rr - 2r^2\right)^5 \overset{(**)}{\geq} 0\end{array}}$$

Now, $43,008t^3 - 126,400t^2 + 124,192t - 40,192$

$\left(\text{where } t = \frac{R}{r} \overset{\text{Euler}}{\geq} 2 \text{ and comma represents the thousands} - \text{separator}\right)$

$= (t - 2)\left(43,008t^2 - 40,384t + 43,424\right) + 46,656 > 0$

$\Rightarrow 43,008R^3 - 126,400R^2r + 124,192Rr^2 - 40,192r^3 > 0$ and

$209,152t^4 - 855,808t^3 + 1,311,424t^2 - 877,440t + 214,794$

$$= (t-2)\left((t-2)(209,152t^2 - 19,200t + 398,016) + 791,424\right) + 205,578 > 0$$
$$\Rightarrow 209,152R^4 - 855,808R^3r + 1,311,424R^2r^2 - 877,440Rr^3 + 214,794r^4 > 0$$
$$\therefore \text{via Gerretsen}, 6(s^2 - 16Rr + 5r^2)^6 + (264Rr - 246r^2)(s^2 - 16Rr + 5r^2)^5$$
$$+r^2(4704R^2 - 8952Rr + 4276r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+r^3(43,008R^3 - 126,400R^2r + 124,192Rr^2 - 40,192r^3)(s^2 - 16Rr + 5r^2)^3$$
$$+r^4\begin{pmatrix}209,152R^4 - 855,808R^3r + 1,311,424R^2r^2\\ -877,440Rr^3 + 214,794r^4\end{pmatrix}(s^2 - 16Rr + 5r^2)^2 \overset{(\bullet)}{\geq} 0$$
$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \geq$ LHS of $(\bullet)$
$$\Leftrightarrow \begin{pmatrix}247,808R^5 - 1,361,664R^4r + 2,948,224R^3r^2\\ -3,097,408R^2r^3 + 1,573,116Rr^4 - 308,477r^5\end{pmatrix}s^2$$
$$\overset{(***)}{\geq} r\begin{pmatrix}3,760,128R^6 - 21,463,040R^5r + 49,317,504R^4r^2\\ -57,311,104R^3r^3 + 35,056,912R^2r^4 - 10,519,128Rr^5 + 1,171,433r^6\end{pmatrix}$$
Now, $247,808t^5 - 1,361,664t^4 + 2,948,224t^3 - 3,097,408t^2 + 1,573,116t - 308,477$
$$= (t-2)\left((t-2)\begin{pmatrix}247,808t^3 - 370,432t^2\\ +475,264t + 285,376\end{pmatrix} + 813,564\right) + 177,147 > 0$$
$$\Rightarrow 247,808R^5 - 1,361,664R^4r + 2,948,224R^3r^2 - 3,097,408R^2r^3 + 1,573,116Rr^4$$
$$-308,477r^5 > 0 \therefore \text{LHS of } (***) \overset{\text{Rouche}}{\geq}$$
$$\begin{pmatrix}247,808R^5 - 1,361,664R^4r + 2,948,224R^3r^2\\ -3,097,408R^2r^3 + 1,573,116Rr^4 - 308,477r^5\end{pmatrix}\begin{pmatrix}2R^2 + 10Rr - r^2\\ -2(R-2r)\sqrt{R^2 - 2Rr}\end{pmatrix}$$
$$\overset{?}{\geq} \text{RHS of } (***)$$
$$\Leftrightarrow 2(R-2r)\begin{pmatrix}247,808R^6 - 1,507,072R^5r + 3,733,376R^4r^2\\ -4,867,456R^3r^3 + 3,532,604R^2r^4 - 1,357,441Rr^5 + 215,739r^6\end{pmatrix}$$
$$\underset{(****)}{\overset{?}{\geq}} 2(R-2r)\sqrt{R^2 - 2Rr}\begin{pmatrix}247,808R^5 - 1,361,664R^4r + 2,948,224R^3r^2\\ -3,097,408R^2r^3 + 1,573,116Rr^4 - 308,477r^5\end{pmatrix}$$
Again, $247,808t^6 - 1,507,072t^5 + 3,733,376t^4 - 4,867,456t^3 + 3,532,604t^2 - 1,357,441t + 215,739$
$$= (t-2)\left((t-2)\left((t-2)\begin{pmatrix}247,808t^3 - 20,224t^2\\ +638,336t + 1,187,712\end{pmatrix} + 2,837,052\right) + 844,911\right)$$
$$+59,049 > 0 \Rightarrow 247,808R^6 - 1,507,072R^5r + 3,733,376R^4r^2 - 4,867,456R^3r^3$$
$$+3,532,604R^2r^4 - 1,357,441Rr^5 + 215,739r^6 > 0 \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0$$
$\therefore$ in order to prove $(****)$, it suffices to prove :
$$\begin{pmatrix}247,808R^6 - 1,507,072R^5r + 3,733,376R^4r^2 -\\ 4,867,456R^3r^3 + 3,532,604R^2r^4 - 1,357,441Rr^5 + 215,739r^6\end{pmatrix}^2$$
$$> (R^2 - 2Rr)\begin{pmatrix}247,808R^5 - 1,361,664R^4r +\\ 2,948,224R^3r^2 - 3,097,408R^2r^3 + 1,573,116Rr^4 - 308,477r^5\end{pmatrix}^2$$
$$\Leftrightarrow 50,751,078,400t^{11} - 543,453,872,128t^{10} + 2,529,415,659,520t^9$$
$$-6,675,031,293,952t^8 + 10,849,957,036,032t^7 - 10,843,569,209,344t^6$$
$$+5,808,184,095,232t^5 - 209,088,406,656t^4 - 1,948,958,574,656t^3$$
$$+1,330,648,500,336t^2 - 395,389,808,740t + 46,543,316,121 > 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)\left((t-2)\left(\begin{matrix}(t-2)(\lambda)+\\7,520,970,248,064\end{matrix}\right)+2,467,147,211,712\right)+441,062,017,776\right)+37,020,180,060\right)+3,486,784,401>0$$

$$\left(\begin{matrix}\text{where } \lambda = 50,751,078,400t^6 - 35,943,088,128t^5 + 139,941,642,240t^4 + \\ 222,194,925,568t^3 + 538,707,279,872t^2 + 1,350,519,504,896t + 3,195,171,795,456\end{matrix}\right)$$

$$\rightarrow \text{true} \because t \geq 2 \text{ and } \therefore \lambda > 0 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore 9(a^k + b^k + c^k) + ab + bc + ca \leq 27 + 3abc \text{ for } k = \frac{1}{2}\ \forall\, a,b,c > 0 \,\Big|\, \sum_{cyc} a = 3$$

$$\text{Also, } 9(a^k + b^k + c^k) + ab + bc + ca - 27 - 3abc$$

$$= 9\sum_{cyc}(1 + (a-1))^k + \sum_{cyc} ab - 27 - 3abc \overset{via \text{ Bernoulli} \because k \geq 1}{\geq}$$

$$9\left(3 + k\left(\sum_{cyc} a - 3\right)\right) + \sum_{cyc} ab - 27 - 3abc \overset{a+b+c=3}{=}$$

$$\sum_{cyc} ab - 3abc \overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2} - 3abc \overset{?}{\geq} 0 \Leftrightarrow a^2b^2c^2 \overset{?}{\geq} a^3b^3c^3 \Leftrightarrow abc \overset{?}{\leq} 1$$

$$\rightarrow \text{true} \because 3 = \sum_{cyc} a \overset{A-G}{\geq} 3\sqrt[3]{abc} \Rightarrow abc \leq 1$$

$$\therefore 9(a^k + b^k + c^k) + ab + bc + ca \geq 27 + 3abc \text{ with } k \geq 1\ \forall\, a,b,c > 0 \,\Big|\, \sum_{cyc} a = 3$$

$\therefore$ combining *all* cases,

$$a)\ 9(a^k + b^k + c^k) + ab + bc + ca \leq 27 + 3abc \text{ with } k = \frac{1}{2}$$

$$\forall\, a,b,c \geq 0 \,\big|\, a+b+c = 3, \text{"="} \text{ iff } a = b = c = 1$$

$$b)\ 9(a^k + b^k + c^k) + ab + bc + ca \geq 27 + 3abc \text{ with } k \geq 1$$

$$\forall\, a,b,c \geq 0 \,\big|\, a+b+c = 3, \text{"="} \text{ iff } (a = b = c = 1) \text{ or, } (a = 3, b = c = 0; k = 1)$$

$$\text{or, } (b = 3, c = a = 0; k = 1) \text{ or, } (c = 3, a = b = 0; k = 1)\ (\text{QED})$$