ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0$ such that $a+b+c=4$. Prove that :

$$\text{(i) } a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} + 2\left(16-9\sqrt{3}\right) \geq \left(6-3\sqrt{3}\right)(ab+bc+ca) \text{ and}$$

$$\text{(ii) } a^{2023} + b^{2023} + c^{2023} + 2\left(16-9\sqrt{3}\right) \geq \left(6-3\sqrt{3}\right)(ab+bc+ca)$$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

We shall first prove :

$$\text{(i) } a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} + 2\left(16-9\sqrt{3}\right) \geq \left(6-3\sqrt{3}\right)(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=4)$ and then : $\text{LHS}-\text{RHS} = 4^{\frac{2024}{2025}} + 2\left(16-9\sqrt{3}\right) > 0 \therefore$ (i) is true (strict inequality)

$\boxed{\text{Case 2}}$ Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=4)$ and then :

$$\text{LHS}-\text{RHS} = \sqrt[2025]{\underbrace{b.b.b\ldots b}_{2024\text{ terms}}.1} + \sqrt[2025]{\underbrace{c.c.c\ldots c}_{2024\text{ terms}}.1} + 2\left(16-9\sqrt{3}\right)$$

$$-16\left(\frac{6-3\sqrt{3}}{2}\right)\left(\frac{2bc}{(b+c)^2}\right) \overset{G-H}{\geq}$$

$$\frac{2025}{\frac{2024}{b}+1} + \frac{2025}{\frac{2024}{c}+1} + 2\left(16-9\sqrt{3}\right) - 8\left(6-3\sqrt{3}\right)\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$= \frac{2025b^2}{2024b+b^2} + \frac{2025c^2}{2024c+c^2} + 32 - 48\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-18\sqrt{3} + 24\sqrt{3}\left(\frac{2bc}{b^2+c^2+2bc}\right) \overset{\substack{\text{Bergstrom} \\ \text{and} \\ \because b+c=4}}{\geq}$$

$$\frac{2025(b+c)^2}{506(b+c)^2+b^2+c^2} + 32 - 48\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$-18\sqrt{3} + 24\sqrt{3}\left(\frac{2bc}{b^2+c^2+2bc}\right)$$

$$= \frac{2025(x+y)}{506(x+y)+x} + 32 - 48\left(\frac{y}{x+y}\right) - 18\sqrt{3} + 24\sqrt{3}\left(\frac{y}{x+y}\right)$$

$$\left(x=b^2+c^2; y=2bc\right) = \frac{18249x^2+12130xy-6071y^2}{(x+y)(507x+506y)} - \frac{6\sqrt{3}(3x-y)}{x+y}$$

$$\overset{6\sqrt{3}<11}{>} \frac{18249x^2+12130xy-6071y^2}{(x+y)(507x+506y)} - \frac{11(3x-y)}{x+y}$$

$$= \frac{1518x^2+1009xy-505y^2}{(x+y)(507x+506y)} = \frac{(506x+505y)(3x-y)}{(x+y)(507x+506y)} > 0 \because x \geq y > 0$$

$\therefore$ (i) is true (strict inequality)

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and then : $\text{LHS}-\text{RHS} \overset{\because a+b+c=4}{=}$

$$\sqrt[2025]{\underbrace{a.a.a\ldots a}_{2024\text{ terms}}.1} + \sqrt[2025]{\underbrace{b.b.b\ldots b}_{2024\text{ terms}}.1} + \sqrt[2025]{\underbrace{c.c.c\ldots c}_{2024\text{ terms}}.1} + 2\left(16-9\sqrt{3}\right)$$

$$-\frac{16(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$\overset{G-H}{\geq}\frac{2025}{\frac{2024}{a}+1}+\frac{2025}{\frac{2024}{b}+1}+\frac{2025}{\frac{2024}{c}+1}+2(16-9\sqrt{3})-\frac{16(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{2025a^2}{2024a+a^2}+\frac{2025b^2}{2024b+b^2}+\frac{2025c^2}{2024c+c^2}+2(16-9\sqrt{3})$$

$$-\frac{16(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \because a+b+c=4}}{\geq}$$

$$\frac{2025(a+b+c)^2}{506(a+b+c)^2+a^2+b^2+c^2}+2(16-9\sqrt{3})-\frac{16(6-3\sqrt{3})(ab+bc+ca)}{(a+b+c)^2}$$

$$=\frac{2025(u+2v)}{506u+1012v+u}+32-\frac{96v}{u+2v}-18\sqrt{3}+48\sqrt{3}\left(\frac{v}{u+2v}\right)$$

$$\left(u=\sum_{cyc}a^2,v=\sum_{cyc}ab\right)=\frac{18249u^2+24260uv-24284v^2}{(u+2v)(507u+1012v)}-\frac{6\sqrt{3}(3u-2v)}{u+2v}$$

$$\overset{6\sqrt{3}<11}{>}\frac{18249u^2+24260uv-24284v^2}{(u+2v)(507u+1012v)}-\frac{11(3u-2v)}{u+2v}$$

$$=\frac{1518u^2+2018uv-2020v^2}{(u+2v)(507u+1012v)}=\frac{2(3u-2v)(253u+505v)}{(u+2v)(507u+1012v)}>0\because u\geq v>0$$

$\therefore$ (i) is true (strict inequality) $\therefore$ combining all cases,

(i) is true (strict inequality) $\forall\, a,b,c\geq 0 \mid a+b+c=4$

We shall now prove :

$$\text{(ii) } a^{2023}+b^{2023}+c^{2023}+2(16-9\sqrt{3})\geq(6-3\sqrt{3})(ab+bc+ca)$$

Case 1 Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$ $(a=4)$ and then : $\text{LHS}-\text{RHS}=4^{2023}+2(16-9\sqrt{3})>0$ $\therefore$ (ii) is true (strict inequality)

Case 2 Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $(b+c=4)$

and then : $\text{LHS}-\text{RHS}=b^{2023}+c^{2023}+2(16-9\sqrt{3})-(6-3\sqrt{3})(bc)$

$$\overset{\substack{\text{Holder}\\ \text{and}\\ A-G}}{\geq}\frac{1}{2^{2022}}(b+c)^{2023}+2(16-9\sqrt{3})-\frac{(6-3\sqrt{3})}{4}(b+c)^2\overset{b+c=4}{=}$$

$$2^{2024}+2(16-9\sqrt{3})-4(6-3\sqrt{3})=2^{2024}+8-6\sqrt{3}>0$$

$\therefore$ (ii) is true (strict inequality)

Case 3 $a,b,c>0$ and then : $\text{LHS}-\text{RHS}\overset{\substack{\text{Holder}\\ \text{and}\\ A-G}}{\geq}\frac{1}{3^{2022}}(a+b+c)^{2023}$

$$+2(16-9\sqrt{3})-(2-\sqrt{3})(a+b+c)^2\overset{a+b+c=4}{=}\frac{4^{2023}}{3^{2022}}+2(16-9\sqrt{3})$$

$$-16(2-\sqrt{3})>\left(\frac{4}{3}\right)^{2022}-2\sqrt{3}>\left(\frac{4}{3}\right)^{5}-4=4\left(\frac{256}{243}-1\right)>0$$

$\therefore$ (ii) is true (strict inequality) $\therefore$ combining all cases,

(ii) is true (strict inequality) $\forall\, a,b,c\geq 0 \mid a+b+c=4$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $ab+bc+ca \le \frac{(a+b+c)^2}{3} = \frac{16}{3}$**, so it suffices to prove that**

$$a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} \ge 2\sqrt{3} \quad \text{and} \quad a^{2023} + b^{2023} + c^{2023} \ge 2\sqrt{3}.$$

$i)$ **Since** $\frac{a}{4}, \frac{b}{4}, \frac{c}{4} \le 1$**, then** $\left(\frac{a}{4}\right)^{\frac{2024}{2025}} \ge \frac{a}{4},\ \left(\frac{b}{4}\right)^{\frac{2024}{2025}} \ge \frac{b}{4},\ \left(\frac{c}{4}\right)^{\frac{2024}{2025}} \ge \frac{c}{4}$**, thus**

$$a^{\frac{2024}{2025}} + b^{\frac{2024}{2025}} + c^{\frac{2024}{2025}} \ge 4^{\frac{2024}{2025}}.\frac{a+b+c}{4} = 4^{\frac{2024}{2025}} = 2.2^{1-\frac{2}{2025}} > 2.2^{1-\frac{1}{5}} = 2\sqrt[5]{16} > 2\sqrt{3},$$

which completes the proof of $i)$.

$ii)$ **By Power Mean inequality, we have**

$$a^{2023} + b^{2023} + c^{2023} \ge 3\left(\frac{a+b+c}{3}\right)^{2023} = \frac{4^{2023}}{3^{2022}} = \frac{2^{4046}}{\sqrt{3}^{4044}} > 2.\frac{\sqrt{3}^{4045}}{\sqrt{3}^{4044}} = 2\sqrt{3},$$

which completes the proof of $ii)$.