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Let $a,b,c\ge 0$ such that $a+b+c=3$. Prove that

$$a)\ 9\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+ab+bc+ca\le 27+3abc.$$

$$b)\ 9\left(a^k+b^k+c^k\right)+ab+bc+ca\ge 27+3abc,\quad \text{with } k\ge 1.$$

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$a)$ Let $x:=\sqrt{a},\ \ y:=\sqrt{b},\ \ z:=\sqrt{c}$ and $p:=x+y+z,\ \ q:=xy+yz+zx,\ \ r:=xyz.$
The given condition is equivalent to $x^2+y^2+z^2=p^2-2q$
$=3,$ and the problem becomes,

$$9(x+y+z)+x^2y^2+y^2z^2+z^2x^2\le 27+3x^2y^2z^2\ \Leftrightarrow\ 9p+q^2-2pr\le 27+3r^2$$

$$\overset{2q=p^2-3}{\Leftrightarrow}\quad 36p+(p^2-3)^2-8pr\le 108+12r^2$$

$$\Leftrightarrow\ 99-36p+6p^2-p^4+8pr+12r^2\ge 0\ \ (*)$$

We have $3=x^2+y^2+z^2\le p^2\le 3(x^2+y^2+z^2)=9,$ then $\sqrt{3}\le p\le 3.$

▪If $\sqrt{3}\le p\le\sqrt{6},$ we have $LHS_{(*)}\overset{r\ge 0}{\ge}9(11-4p)+p^2(6-p^2)\ge 0.$

▪If $\sqrt{3}\le p\le 3,$ we have

$$0\le (a-b)^2(b-c)^2(c-a)^2=\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}$$

$$\Rightarrow\ r\ge\frac{-2p^3+9pq-2\sqrt{(p^2-3q)^3}}{27}\overset{2q=p^2-3}{=}\frac{5p^3-27p-\sqrt{2(9-p^2)^3}}{54}.$$

$$\Rightarrow\ LHS_{(*)}\ge 105-36p+3p^2-\frac{31p^4}{27}+\frac{23p^6}{243}-\frac{2p(5p^2-9)\sqrt{2(9-p^2)^3}}{243}$$

$$=(3-p)\left(35-\frac{p}{3}+\frac{8p^2}{9}+\frac{8p^3}{27}-\frac{23p^4}{81}-\frac{23p^5}{243}-\frac{2p(5p^2-9)\sqrt{2(3+p)^3(3-x)}}{243}\right)$$

$$\overset{AM-GM}{\ge}\ (3-p)\left(35-\frac{p}{3}+\frac{8p^2}{9}+\frac{8p^3}{27}-\frac{23p^4}{81}-\frac{23p^5}{243}\right.$$
$$\left.-\frac{p(5p^2-9)[2(3+p)^2(3-p)+(3+p)]}{243}\right)$$

$$=(3-p)\left[(3-p)\left[\frac{35}{3}+\frac{121p}{27}+(3-p)\left(\frac{164p^2}{243}+\frac{239p^3}{729}+\frac{44p^4}{729}\right)+\frac{46p^5}{2187}\right]\right.$$
$$\left.+\frac{4p^6}{2187}\right]\overset{3\ge p}{\ge}0,$$

which completes the proof of $a)$. Equality holds iff $a=b=c=1.$
$b)$ Since $k\ge 1$ then by Power Mean inequality, we have

$$a^k+b^k+c^k\ge 3\left(\frac{a+b+c}{3}\right)^k=3,$$

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and by AM − GM inequality, we have

$$\boldsymbol{ab+bc+ca=\frac{(a+b+c)(ab+bc+ca)}{3}\geq 3abc.}$$

Using these two inequalities, we have

$$\boldsymbol{9\left(a^k+b^k+c^k\right)+ab+bc+ca\geq 27+3abc.}$$

So the proof is complete. Equality holds iff

$$\boldsymbol{a=b=c=1 \text{ or } k=1, a=3, b=c=0 \text{ and permutations.}}$$