ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c \geq 0$ and $a^2+b^2+c^2 \neq 0$. Prove that :

$$\sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} \geq \frac{a+b+c}{2}$$

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Case 1 : Exactly one variable $= 0$ and WLOG we may assume $a=0$

$$\text{and then : } \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} = \frac{b^3+c^3}{b^2+c^2} \overset{\text{Chebyshev}}{\geq} \frac{\frac{1}{2}(b+c)(b^2+c^2)}{b^2+c^2} = \frac{b+c}{2}$$

$$\therefore \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} \geq \frac{a+b+c}{2}$$

Case 2 : Exactly two variables $= 0$ and WLOG we may assume $b=c=0$

$$\text{and then : } \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} = \frac{a^3}{a^2} = a > \frac{a}{2}$$

$$\Rightarrow \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc} > \frac{a+b+c}{2}$$

Case 3 : $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z$

$$\Rightarrow x+y-z=2c>0, y+z-x=2a>0 \text{ and } z+x-y=2b>0$$

$$\Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z \text{ form sides of } a \text{ triangle}$$

with semiperimeter, circumradius and inradius $=$ s, R, r (say) yielding $2\sum_{cyc} a =$

$$\sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a=s-x, b=s-y, c=s-z \therefore abc = r^2 s \rightarrow (2)$$

$$\text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr+r^2 \rightarrow (3),$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4),$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \rightarrow (5),$$

$$\sum_{cyc} a^3 = \left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) \overset{\text{via (1)}}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} a^3 = s(s^2 - 12Rr) \to (6) \text{ and } \sum_{cyc} a^4 = \left(\sum_{cyc} a^2\right)^2 - 2\sum_{cyc} a^2b^2 \overset{\text{via (4) and (5)}}{=}$$

$$(s^2 - 8Rr - 2r^2)^2 - 2r^2((4R+r)^2 - 2s^2)$$

$$\Rightarrow \sum_{cyc} a^4 = (s^2 - 8Rr - 2r^2)^2 - 2r^2((4R+r)^2 - 2s^2) \to (7)$$

$$\text{Now, } \sum_{cyc} \frac{a(a^2+3bc)}{(b+c)^2 + a^2 + 3bc} = \sum_{cyc} \frac{a^3}{\sum_{cyc} a^2 + 5bc} + 3abc\sum_{cyc} \frac{1}{\sum_{cyc} a^2 + 5bc} \to (\bullet)$$

$$\text{Firstly, } \sum_{cyc}\left(a^3\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= \sum_{cyc}\left(a^3\left(\left(\sum_{cyc} a^2\right)^2 + 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab - bc\right) + 25abc.a\right)\right)$$

$$= \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^2\right)^2 + 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^3\right) - 5abc\left(\sum_{cyc} a^2\right)^2$$

$$+25abc\sum_{cyc} a^4 \overset{\text{via (2),(3),(4),(6) and (7)}}{=}$$

$$s(s^2-12Rr)(s^2-8Rr-2r^2)^2 + 5(s^2-8Rr-2r^2)(4Rr+r^2).s(s^2-12Rr)$$

$$-5r^2s(s^2-8Rr-2r^2)^2 + 25r^2s\left((s^2-8Rr-2r^2)^2 - 2r^2((4R+r)^2-2s^2)\right)$$

$$\therefore \sum_{cyc}\left(a^3\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= s\begin{pmatrix} s^6 - (8Rr - 21r^2)s^4 - (144R^2 + 380Rr - 14r^2)r^2s^2 \\ +r^3(1152R^3 + 1056R^2r + 312Rr^2 + 30r^3)\end{pmatrix} \to (\text{i})$$

$$\text{Secondly, } \sum_{cyc}\left(\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right)$$

$$= 3\left(\sum_{cyc} a^2\right)^2 + 10\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) + 25abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2),(3) and (4)}}{=} 3(s^2-8Rr-2r^2)^2 + 10(4Rr+r^2)(s^2-8Rr-2r^2) + 25r^2s^2$$

$$\therefore \sum_{cyc}\left(\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)\right) = 3s^4 - (8Rr - 23r^2)s^2 - 8r^2(4R+r)^2 \to (\text{ii})$$

$$\text{Thirdly, } \left(\sum_{cyc} a^2 + 5bc\right)\left(\sum_{cyc} a^2 + 5ca\right)\left(\sum_{cyc} a^2 + 5ab\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+5\left(\sum_{cyc}a^2\right)^2\left(\sum_{cyc}ab\right)+25abc\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)+125(abc)^2$$

$$\overset{\text{via }(1),(2),(3)\text{ and }(4)}{=}(s^2-8Rr-2r^2)^3+5(4Rr+r^2)(s^2-8Rr-2r^2)^2$$
$$+25r^2s^2(s^2-8Rr-2r^2)+125r^4s^2$$

$$\therefore\left(\sum_{cyc}a^2+5bc\right)\left(\sum_{cyc}a^2+5ca\right)\left(\sum_{cyc}a^2+5ab\right)$$
$$=s^6-(4Rr-24r^2)s^4-(128R^2+264Rr-67r^2)r^2s^2+12r^3(4R+r)^3\to\text{(iii)}$$

$$\text{Now, }(\bullet),\text{(i)},\text{(ii)},\text{(iii)}\Rightarrow\sum_{cyc}\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\ge\frac{a+b+c}{2}$$

$$\Leftrightarrow s.\frac{s^6-(8Rr-21r^2)s^4-(144R^2+380Rr-14r^2)r^2s^2+r^3(1152R^3+1056R^2r+312Rr^2+30r^3)}{s^6-(4Rr-24r^2)s^4-(128R^2+264Rr-67r^2)r^2s^2+12r^3(4R+r)^3}$$
$$+3r^2s.\frac{3s^4-(8Rr-23r^2)s^2-8r^2(4R+r)^2}{s^6-(4Rr-24r^2)s^4-(128R^2+264Rr-67r^2)r^2s^2+12r^3(4R+r)^3}\ge\frac{s}{2}$$

$$\Leftrightarrow\boxed{s^6-(12Rr-36r^2)s^4-(160R^2+544Rr-99r^2)r^2s^2+96Rr^3(4R+r)^2\overset{(*)}{\ge}0}$$

*and* $\because(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of }(*)\ge(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow\boxed{\begin{array}{c}(36Rr+21r^2)s^4-(928R^2+64Rr-24r^2)r^2s^2\\+r^3(5632R^3-3072R^2r+1296Rr^2-125r^3)\overset{(**)}{\ge}0\end{array}}\ \textit{and}$$

$\because(36Rr+21r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**)\ge(36Rr+21r^2)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow\boxed{(112R^2+124Rr-93r^2)s^2\overset{(***)}{\ge}r(1792R^3+1344R^2r-1878Rr^2+325r^3)}$$

$$\textit{Again},(112R^2+124Rr-93r^2)s^2\overset{\text{Gerretsen}}{\ge}$$
$$(112R^2+124Rr-93r^2)(16Rr-5r^2)\overset{?}{\ge}$$
$$r(1792R^3+1344R^2r-1878Rr^2+325r^3)\Leftrightarrow 10r(8R^2-23Rr+14r^2)\overset{?}{\ge}0$$
$$\Leftrightarrow 10r(R-2r)(8R-7r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\ge\frac{a+b+c}{2}\ \therefore\text{combining }\textit{all}\text{ cases},$$

$$\sum_{cyc}\frac{a(a^2+3bc)}{(b+c)^2+a^2+3bc}\ge\frac{a+b+c}{2}\ \forall\,a,b,c\ge0\text{ and }a^2+b^2+c^2\ne0,$$

$''=''$ iff $(a=0,b=c>0)$ or $(b=0,c=a>0)$ or $(c=0,a=b>0)$
or $(a=b=c>0)$ (QED)