ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c, d \geq 0$ such that $a^2 + b^2 + c^2 + d^2 = 2023$.

Prove that : $$\frac{1}{a^4+2}+\frac{1}{b^4+2}+\frac{1}{c^4+2}+\frac{1}{d^4+2}\geq\frac{64}{4092561}$$

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We first prove that : $$\frac{1}{x^2+2}\geq\frac{1}{\left(\frac{2023}{4}\right)^2+2}-\left(x-\frac{2023}{4}\right).\frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2}$$

$$\forall\, x\geq 0 \Leftrightarrow \frac{1}{x^2+2}-\frac{16}{4092561}+\frac{64736(4x-2023)}{4092561^2}\geq 0$$

$$\Leftrightarrow \frac{2023^2-16x^2}{4092561(x^2+2)}-\frac{64736(2023-4x)}{4092561^2}\geq 0$$

$$\Leftrightarrow \frac{2023-4x}{4092561}\left(\frac{2023+4x}{x^2+2}-\frac{32.2023}{2023^2+32}\right)\geq 0$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right).\frac{(m+4x)(m^2+32)-32m(x^2+2)}{(m^2+32)(x^2+2)}\geq 0\ (m=2023)$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right).\frac{m^3+4m^2x-32mx^2-32m+128x}{(m^2+32)(x^2+2)}\geq 0$$

$$\Leftrightarrow \left(\frac{m-4x}{4092561}\right).\frac{(m-4x)(m^2-32+8mx)}{(m^2+32)(x^2+2)}\geq 0$$

$$\Leftrightarrow \frac{(2023-4x)^2(4092497+8.2023x)}{4092561(2023^2+32)(x^2+2)}\geq 0 \to \text{true} \because x\geq 0$$

$$\therefore \frac{1}{x^2+2}\geq\frac{1}{\left(\frac{2023}{4}\right)^2+2}-\left(x-\frac{2023}{4}\right)\frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2}\overset{x\equiv a^2,b^2,c^2,d^2}{\Rightarrow}$$

$$\frac{1}{a^4+2}\geq\frac{1}{\left(\frac{2023}{4}\right)^2+2}-\left(a^2-\frac{2023}{4}\right).\frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2}\ \text{and analogs}$$

$$\overset{\text{summation}}{\Rightarrow}\frac{1}{a^4+2}+\frac{1}{b^4+2}+\frac{1}{c^4+2}+\frac{1}{d^4+2}\geq$$

$$\frac{4}{\left(\frac{2023}{4}\right)^2+2}-\frac{\frac{2023}{2}}{\left(\left(\frac{2023}{4}\right)^2+2\right)^2}.\left(\begin{matrix}a^2+b^2+\\c^2+d^2-2023\end{matrix}\right)\overset{a^2+b^2+c^2+d^2=2023}{=}\frac{4}{\left(\frac{2023}{4}\right)^2+2}$$

$$=\frac{64}{4092561}\therefore\frac{1}{a^4+2}+\frac{1}{b^4+2}+\frac{1}{c^4+2}+\frac{1}{d^4+2}\geq\frac{64}{4092561}$$

$$\forall\, a,b,c,d\geq 0\,\middle|\,a^2+b^2+c^2+d^2=2023,''=''\ \text{iff}\ a=b=c=d=\frac{\sqrt{2023}}{2}\ \text{(QED)}$$