ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $2,25.\left(1+\ln^2\left(\frac{4(ab+bc+ca)}{3}\right)\right)\geq(a+b+c)^2$, then :

$$\frac{(2(a+b)^{a+b}-1)bc}{b+c}+\frac{(2(b+c)^{b+c}-1)ca}{c+a}+\frac{(2(c+a)^{c+a}-1)ab}{a+b}\geq\frac{8abc(a+b+c)^2}{3}$$

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Let $x:=\frac{4(ab+bc+ca)}{3}$. Since $(a+b+c)^2\geq 3(ab+bc+ca)=2,25.x$, then

$1+\ln^2(x)\geq x$. This inequality is true for $x\leq 1$. If $x>1$,

let $f(t)=\ln(t)-\sqrt{t-1}, t\geq 1$. We have

$f'(t)=\frac{1}{t}-\frac{1}{2\sqrt{t-1}}=-\frac{\left(\sqrt{t-1}-1\right)^2}{2t\sqrt{t-1}}\leq 0,\ \ \forall t>1$, then f is strictly decreasing, hence

$f(x)<f(1)=0$ or $1+\ln^2(x)<x$, which is not true.

Therefore, $x\leq 1$ or $ab+bc+ca\leq\frac{3}{4}$ (1)

Now, let us prove that for all $t>0$, $2t^t\geq t^2+1$ (2).

Let $g(t)=t.\ln t-\ln\left(\frac{t^2+1}{2}\right),\ \ t>0.$

We have $g'(t)=\ln t+1-\frac{2t}{t^2+1}$ and $g''(t)=\frac{t^4+2t^3+t^2+(t-1)^2}{t(t^2+1)^2}\geq 0$, then g' is increasing and since $g'(1)=$ 0, then g is decreasing on $(0,1]$ and increasing on $[1,\infty)$, hence

$g(t)\geq g(1)=0,\forall t>0\ \Leftrightarrow\ t.\ln t\geq\ln\left(\frac{t^2+1}{2}\right)$, then $2t^t\geq t^2+1,\forall t>0$,

equality for $x=1$. Now, we have

$$\sum_{cyc}\frac{(2(a+b)^{a+b}-1)bc}{b+c}\overset{(2)}{\geq}\sum_{cyc}\frac{(a+b)^2bc}{b+c}=abc\sum_{cyc}\frac{(a+b)^2}{ab+ca}\overset{CBS}{\geq}\frac{4abc(a+b+c)^2}{2(ab+bc+ca)}$$

$$\overset{(1)}{\geq} \frac{8abc(a+b+c)^2}{3},$$

as desired. Equality holds iff $a = b = c = \frac{1}{2}$.