ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ such that $e^{1-a-b-c}\geq(a+b+c)^{a+b+c-2}$, then

$$\left(\left(\frac{c^{1-c}}{ab}\right)^3+\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3\geq 129140163$$

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$$\text{Let } f(x)=(x-2).\ln x+x-1,\ \text{ for } x>0.$$

$$\text{We have } f'(x)=\ln x+\frac{2(x-1)}{x},$$

$$\forall x>0, \text{then } f \text{ is decreasing on } (0,1] \text{ and increasing on } [1,\infty),$$

hence, $f(x)\geq f(1)=0,\ \forall x>0,$ or $e^{1-x}\leq x^{x-2},\ \forall x>0,$ with equality for $x=1$.

$$\text{But we have } e^{1-(a+b+c)}\geq(a+b+c)^{(a+b+c)-2}, \text{then}$$

$$e^{1-(a+b+c)}=(a+b+c)^{(a+b+c)-2}. \text{Therefore, } a+b+c=1.$$

Now, by Hölder's inequality, we have

$$\left(\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3+\left(\frac{c^{1-c}}{ab}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3.(b+c+a)^6.(c+a+b)^6\geq$$

$$\geq\left(\sqrt[17]{\left(\frac{a^{1-a}}{bc}\right)^6.(a^{2a-2})^3.b^6.c^6}+\sqrt[17]{\left(\frac{b^{1-b}}{ca}\right)^6(b^{2b-2})^3c^6a^6}+\sqrt[17]{\left(\frac{c^{1-c}}{ab}\right)^6(c^{2c-2})^3a^6b^6}\right)^{17}=3^{17}$$

Therefore

$$\left(\left(\frac{c^{1-c}}{ab}\right)^3+\left(\frac{a^{1-a}}{bc}\right)^3+\left(\frac{b^{1-b}}{ca}\right)^3\right)^2.\left(a^{2a-2}+b^{2b-2}+c^{2c-2}\right)^3\geq 3^{17}=129140163.$$

Equality holds iff $a=b=c=\frac{1}{3}$.