ROMANIAN MATHEMATICAL MAGAZINE

Let $a,b,c>0$, prove that

$$\left(1+\frac{2}{b+c}\right)^{(b+c-2a)^3}.\left(1+\frac{2}{c+a}\right)^{(c+a-2b)^3}.\left(1+\frac{2}{a+b}\right)^{(a+b-2c)^3}.e^{-20\left(a^2+b^2+c^2-ab-bc-ca\right)}\le 1$$

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The desired inequality is equivalent to

$$(b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)+(c+a-2b)^3\ln\left(1+\frac{2}{c+a}\right)+(a+b-2c)^3\ln\left(1+\frac{2}{a+b}\right)\le$$

$$\le 20\left(a^2+b^2+c^2-ab-bc-ca\right).$$

Using te known inequality $0\le\ln(1+x)\le x,\ \ \forall x\ge 0$, we have

$$(b+c-2a)\ln\left(1+\frac{2}{b+c}\right)=(b+c)\ln\left(1+\frac{2}{b+c}\right)-2a\ln\left(1+\frac{2}{b+c}\right)\le$$

$$\le(b+c).\frac{2}{b+c}-0=2$$

$$\Rightarrow (b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)\le 2(b+c-2a)^2\ \ \text{(and analogs)}.$$

Therefore

$$\sum_{cyc}(b+c-2a)^3\ln\left(1+\frac{2}{b+c}\right)\le 2\sum_{cyc}(b+c-2a)^2$$

$$=2\sum_{cyc}\left(4a^2+b^2+c^2-4ab+2bc-4ca\right)$$

$$=12\left(a^2+b^2+c^2-ab-bc-ca\right)\le 20\left(a^2+b^2+c^2-ab-bc-ca\right),$$

as desired. Equality holds iff $a=b=c$.