ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then:

$$(a+b+c)\left(\frac{1}{2a+3b}+\frac{1}{2b+3c}+\frac{1}{2c+3a}\right)+\frac{\sqrt[4]{(1+a)(1+b)(1+c)(1+\sqrt[3]{abc})}}{\sqrt[3]{(1+\sqrt{ab})(1+\sqrt{bc})(1+\sqrt{ca})}}\geq\frac{14}{5}$$

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By CBS inequality, we have

$$(a+b+c)\left(\frac{1}{2a+3b}+\frac{1}{2b+3c}+\frac{1}{2c+3b}\right)\geq\frac{(a+b+c).3^2}{(2a+3b)+(2b+3c)+(2c+3b)}=\frac{9}{5}.$$

So it suffices to prove that

$$\sqrt[4]{(1+a)(1+b)(1+c)(1+\sqrt[3]{abc})}\geq\sqrt[3]{(1+\sqrt{ab})(1+\sqrt{bc})(1+\sqrt{ca})}$$

$$\Leftrightarrow\ 3\left(\sum_{cyc}\ln(1+a)+\ln\left(1+\sqrt[3]{abc}\right)\right)\geq 4\sum_{cyc}\ln\left(1+\sqrt{ab}\right)\ \ (1)$$

Let $a=e^x, b=e^y, c=e^z,\ \ x,y,z\in\mathbb{R}$, and let $f(t)=\ln(1+e^t)$, $t\in\mathbb{R}$. We have

$$(1)\Leftrightarrow 3\big(f(x)+f(y)+f(z)\big)+3f\left(\frac{x+y+z}{3}\right)\geq 4\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right].$$

We have

$f''(t)=\dfrac{e^t}{(1+e^t)^2}>0$, so f is convex, and by Popoviciu and Jensen inequalities, we get

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right)\geq 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right]$$

$$f(x)+f(y)\geq 2f\left(\frac{x+y}{2}\right),\ \ f(y)+f(z)\geq 2f\left(\frac{y+z}{2}\right),\ \ f(z)+f(x)\geq 2f\left(\frac{z+x}{2}\right).$$

Adding these inequalities yields the desiredd inequality (1). So the proof is complete.

Equality holds iff $a=b=c$.