ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ *then prove that* :

$$2+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{a+b+c} \le 4\left(\frac{a+b}{3a+3b+2c}+\frac{b+c}{3b+3c+2a}+\frac{c+a}{3c+3a+2b}\right)$$

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$$2+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{a+b+c}-4\left(\frac{a+b}{3a+3b+2c}+\frac{b+c}{3b+3c+2a}+\frac{c+a}{3c+3a+2b}\right)$$

$$\overset{\text{CBS}}{\le} 2+\frac{\sqrt{3\sum_{cyc}ab}}{\sum_{cyc}a}-4\sum_{cyc}\frac{(b+c)^2}{3(b+c)^2+2a(b+c)} \overset{\text{Bergstrom}}{\le}$$

$$2+\frac{\sqrt{3\sum_{cyc}ab}}{\sum_{cyc}a}-\frac{16\left(\sum_{cyc}a\right)^2}{6\sum_{cyc}a^2+10\sum_{cyc}ab} \overset{?}{\le} 0$$

$$\Leftrightarrow \frac{8(m+2n)-6m-10n}{3m+5n} \overset{?}{\ge} \sqrt{\frac{3n}{m+2n}} \left(m=\sum_{cyc}a^2, n=\sum_{cyc}ab\right)$$

$$\Leftrightarrow (m+2n)(2m+6n)^2 \overset{?}{\ge} 3n(3m+5n)^2 \Leftrightarrow 4m^3+5m^2n-6mn^2-3n^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow (m-n)(4m^2+9mn+3n^2) \overset{?}{\ge} 0 \to \text{true} \because m \ge n$$

$$\therefore 2+\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{a+b+c} \le 4\left(\frac{a+b}{3a+3b+2c}+\frac{b+c}{3b+3c+2a}+\frac{c+a}{3c+3a+2b}\right)$$

$$\forall\, a,b,c>0, ''='' \text{ iff } a=b=c \text{ (QED)}$$