ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b,c>0\Rightarrow$$

$$\frac{(a+b+c)^2}{\ln\left((a+1)^{e^a-1}.(b+1)^{e^b-1}.(c+1)^{e^c-1}\right)}+2\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right)<9$$

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$$\text{We shall first prove that}: (e^x-1).\ln(x+1)>x^2\ \forall\, x>0$$

$$\because e^x-1\ge x+\frac{x^2}{2}\ \forall\, x>0\ \therefore (e^x-1).\ln(x+1)\ge\left(x+\frac{x^2}{2}\right).\ln(x+1)\overset{?}{>}x^2$$

$$\Leftrightarrow \ln(x+1)\underset{(i)}{\overset{?}{>}}\frac{x}{1+\frac{x}{2}}$$

$$\text{Let } f(x)=\ln(x+1)-\frac{x}{1+\frac{x}{2}}\ \forall\, x\ge 0 \text{ and then}: f'(x)=\frac{x^2}{(x+1)(x+2)^2}\ge 0$$

$$\Rightarrow f(x)\text{ is}\uparrow\text{on }[0,\infty)\Rightarrow\forall\, x\ge 0, f(x)\ge f(0)=0\ \therefore\ \forall\, x>0, f(x)>f(0)$$

$$\Rightarrow \text{(i) is true}\ \therefore \boxed{(e^x-1).\ln(x+1)>x^2\ \forall\, x>0}\to(1)$$

$$\text{We shall now prove that}: \sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right)\le 3$$

$$\Leftrightarrow 2\sum_{cyc}x^2-\sum_{cyc}x^3\le 3\left(x=\sqrt{\frac{a}{b}}, y=\sqrt{\frac{b}{c}}, z=\sqrt{\frac{c}{a}}\right)$$

$$\Leftrightarrow \sum_{cyc}x^3+3xyz\ge\left(2\sum_{cyc}x^2\right).\sqrt[3]{xyz}\ (\because xyz=1)$$

$$\Leftrightarrow\left(\sum_{cyc}x^3+3xyz\right)^3\overset{(*)}{\ge}8xyz\left(\sum_{cyc}x^2\right)^3$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ *and* $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form* sides of *a* triangle with semiperimeter, circumradius *and* inradius $=s,R,r$ (*say*)

$$\text{yielding } 2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(1)\Rightarrow x=s-X, y=s-Y,$$

$$z=s-Z\text{ and such substitutions}\Rightarrow\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)$$

$$\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(2),\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via (1) and (2)}}{=}$$

$$s^2-(4Rr+r^2)\Rightarrow\sum_{cyc}x^2=s^2-8Rr-2r^2\to(3)\text{ and}$$

$$\sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x+y)(y+z)(z+x) \overset{via\ (1)}{=} s^3 - 3XYZ = s^3 - 12Rrs$$

$$\Rightarrow \sum_{cyc} x^3 = s(s^2 - 12Rr) \to (4)\ and\ xyz = (s-X)(s-Y)(s-Z) = r^2 s \to (5)$$

$$\therefore via\ (3),(4),(5),(*) \Leftrightarrow \left(s(s^2-12Rr)+3r^2s\right)^3 \geq 8r^2s(s^2-8Rr-2r^2)^3$$

$$\Leftrightarrow s^8 - (36Rr - r^2)s^6 + r^2(432R^2 - 24Rr + 75r^2)s^4$$

$$-r^3(1728R^3 + 240R^2r + 1092Rr^2 + 69r^3)s^2 + 64r^5(4R+r)^3 \overset{(**)}{\geq} 0$$

Now, via Gerretsen, $(s^2-16Rr+5r^2)^4 + (28Rr-19r^2)(s^2-16Rr+5r^2)^3$

$+r^2(240R^2-396Rr+210r^2)(s^2-16Rr+5r^2)^2 \geq 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \geq (s^2-16Rr+5r^2)^4$

$$+(28Rr-19r^2)(s^2-16Rr+5r^2)^3$$

$$+r^2(240R^2-396Rr+210r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (208R^3 - 660R^2r + 792Rr^2 - 311r^3)s^2 \overset{(***)}{\geq}$$

$$r(3072R^4 - 10112R^3r + 12972R^2r^2 - 6492Rr^3 + 859r^4)$$

$$\because 208R^3 - 660R^2r + 792Rr^2 - 311r^3$$

$$= (R-2r)(208R^2 - 244Rr + 304r^2) + 297r^3 \overset{Euler}{\geq} 297r^3 > 0$$

$$\therefore \text{LHS of } (***) \overset{Rouche}{\geq}$$

$$(208R^3 - 660R^2r + 792Rr^2 - 311r^3)\begin{pmatrix} 2R^2 + 10Rr - r^2 \\ -2(R-2r)\sqrt{R^2-2Rr} \end{pmatrix}$$

$$\overset{?}{\geq} r(3072R^4 - 10112R^3r + 12972R^2r^2 - 6492Rr^3 + 859r^4)$$

$$\Leftrightarrow \boxed{\begin{gathered}(R-2r)(208R^4 - 740R^3r + 964R^2r^2 - 579Rr^3 + 137r^4) \underset{(****)}{\overset{?}{\geq}} \\ (R-2r)\sqrt{R^2-2Rr}.\,(208R^3 - 660R^2r + 792Rr^2 - 311r^3)\end{gathered}}$$

$$\because 208R^4 - 740R^3r + 964R^2r^2 - 579Rr^3 + 137r^4$$

$$= (R-2r)(46R^3 + 162R^2(R-2r) + 316Rr^2 + 53r^3) + 243r^3 \overset{Euler}{\geq} 243r^3 > 0$$

$and \because R - 2r \overset{Euler}{\geq} 0 \therefore$ in order to prove $(****)$, it suffices to prove :

$$(208R^4 - 740R^3r + 964R^2r^2 - 579Rr^3 + 137r^4)^2$$

$$> (R^2-2Rr)(208R^3 - 660R^2r + 792Rr^2 - 311r^3)^2$$

$$\Leftrightarrow 53248t^7 - 365568t^6 + 1037376t^5 - 1544208t^4 + 1249120t^3 - 482592t^2$$

$$+34796t + 18769 > 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left((t-2)P + 987552\right) + 335340\right) + 59049 > 0$$

where $P = 30208t^4 + 23040t^3(t-2) + 121920t^2 + 166256t + 414976 \to$ true

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right) \overset{(\bullet)}{\leq} 3}$$

$$\text{Again, } \frac{(a+b+c)^2}{\ln\left((a+1)^{e^{a}-1}.(b+1)^{e^{b}-1}.(c+1)^{e^{c}-1}\right)}$$

$$= \frac{(a+b+c)^2}{(e^a-1).\ln(a+1)+(e^b-1).\ln(b+1)+(e^c-1).\ln(c+1)} \overset{\text{via }(1)}{<} \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

$$\leq 3 \therefore \boxed{\frac{(a+b+c)^2}{\ln\left((a+1)^{e^{a}-1}.(b+1)^{e^{b}-1}.(c+1)^{e^{c}-1}\right)} \overset{(\bullet\bullet)}{<} 3} \therefore (\bullet)+(\bullet\bullet) \Rightarrow$$

$$\frac{(a+b+c)^2}{\ln\left((a+1)^{e^{a}-1}.(b+1)^{e^{b}-1}.(c+1)^{e^{c}-1}\right)} + 2\sum_{cyc}\left(\left(2-\sqrt{\frac{a}{b}}\right)\frac{a}{b}\right) < 9$$

$$\forall\, a,b,c > 0\ (QED)$$