ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \geq 0$, prove that

$$\sqrt[3]{e^{a^2+ab+b^2}}^{8} + \sqrt[3]{e^{a^2+6ab+b^2}} \geq e^{2a^2+2b^2} + e^{(a+b)^2} + e^{ab}\left(e^{3ab} - \sqrt[3]{e^{ab}}\right)$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since the sequence $\left(\frac{8(a^2+ab+b^2)}{3}, \frac{a^2+6ab+b^2}{3}, \frac{4ab}{3}\right)$ majorizes the sequence $(2a^2+2b^2, (a+b)^2, 4ab)$ and since the function $f(x) = e^x, x \geq 0$ is convex, then by Karamata's inequality, we have

$$f\left(\frac{8(a^2+ab+b^2)}{3}\right) + f\left(\frac{a^2+6ab+b^2}{3}\right) + f\left(\frac{4ab}{3}\right)$$
$$\geq f(2a^2+2b^2) + f((a+b)^2) + f(4ab),$$

which is equivalent to

$$\sqrt[3]{e^{a^2+ab+b^2}}^{8} + \sqrt[3]{e^{a^2+6ab+b^2}} \geq e^{2a^2+2b^2} + e^{(a+b)^2} + e^{ab}\left(e^{3ab} - \sqrt[3]{e^{ab}}\right),$$

as desired. Equality holds iff $a = b = 0$.