ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0$ then:

$$\frac{2ab}{a+b}+\sqrt[6]{4(a^3+b^3)}.\sqrt{a+b}\le\frac{3(a+b)}{2}$$

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By AM − GM inequality, we have

$$\sqrt[3]{4(a^3+b^3)(a+b)^3}\le\frac{(a+b)^2+(a+b)^2+4(a^2-ab+b^2)}{3}=2(a^2+b^2).$$

So it suffices to prove that

$$\frac{2ab}{a+b}+\sqrt{2(a^2+b^2)}\le\frac{3(a+b)}{2},$$

which is equivalent to

$$\sqrt{2(a^2+b^2)}-(a+b)\le\frac{a+b}{2}-\frac{2ab}{a+b}\ \text{ or }\ \frac{(a-b)^2}{\sqrt{2(a^2+b^2)}+(a+b)}\le\frac{(a-b)^2}{2(a+b)},$$

which is true by AM − QM inequality : $\sqrt{2(a^2+b^2)}\ge a+b.$

Equality holds iff $a=b$.