If $a,b,c>0$, then :

$$\frac{81abc}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^3}+14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 24$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x$ $=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \rightarrow (1) \Rightarrow a=s-x, b=s-y, c=s-z$

$\therefore abc=r^2s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab=4Rr+r^2 \rightarrow (3) \therefore \frac{81abc}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^3}+14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

$$\overset{\substack{\text{CBS}\\ \text{and}\\ \text{via (2)}}}{\geq} \frac{81r^2s}{\left(\sqrt{3\left(\sum_{cyc} ab\right)}\right)^3}+14\sum_{cyc}\frac{s-x}{x} \overset{\text{via (3)}}{=} \frac{81r^2s}{3(4Rr+r^2).\sqrt{3(4Rr+r^2)}}$$

$$+14\left(\frac{s\sum_{cyc} xy}{xyz}-3\right) \geq \frac{81r^2s}{3(4Rr+r^2).s}+14\left(\frac{s(s^2+4Rr+r^2)}{4Rrs}-3\right)$$

$$\left(\begin{array}{c}\because s^2-12Rr-3r^2=s^2-16Rr+5r^2+4r(R-2r) \overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq} 0\\ \Rightarrow 3(4Rr+r^2)\leq s^2\end{array}\right)$$

$$=\frac{27r}{4R+r}+\frac{7(s^2-8Rr+r^2)}{2Rr} \overset{?}{\geq} 24 \Leftrightarrow \frac{7(4R+r)(s^2-8Rr+r^2)+54Rr^2}{2Rr(4R+r)} \overset{?}{\geq} 24$$

$$\Leftrightarrow (28R+7r)s^2 \underbrace{\overset{?}{\geq}}_{(*)} r(416R^2+22Rr-7r^2)$$

$$\text{Now, } (28R+7r)s^2 \overset{\text{Gerretsen}}{\geq} (28R+7r)(16Rr-5r^2) \overset{?}{\geq} r(416R^2+22Rr-7r^2)$$

$$\Leftrightarrow 16R^2-25Rr-14r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(16R+7r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{81abc}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^3}+14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq 24$$

$\forall a,b,c>0$, " = " iff $a=b=c$ (QED)

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Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM and CBS inequality, we have

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^3 \le 3(ab+bc+ca)(a+b+c).$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)} = \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Using these inequalities, we have

$$\frac{81abc}{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^3}+14\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

$$\ge \frac{27abc}{(ab+bc+ca)(a+b+c)}+\frac{7(a+b+c)^2}{ab+bc+ca}$$

$$= \frac{27abc+7(a+b+c)^3}{(ab+bc+ca)(a+b+c)} \overset{?}{\ge} 24$$

$$\Leftrightarrow 7(a^3+b^3+c^3) \ge 3ab(a+b)+3bc(b+c)+3ca(c+a)+3abc,$$

which is true by AM − GM inequality :

$ab(a+b) \le (a^2+b^2-ab)(a+b) = a^3+b^3$ (and analogs) and $3abc \le a^3+b^3+c^3$.

So the proof is complete. Equality holds iff $a=b=c$.