ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ prove that:

$$\sqrt{\frac{1}{a}+\frac{1}{b}-\frac{2}{a+b+c}}+\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}+\sqrt{\frac{1}{c}+\frac{1}{a}-\frac{2}{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{ab^2+bc^2+ca^2}}{3\sqrt[6]{abc}}\geq 3$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\sum_{cyc}\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}=\sum_{cyc}\sqrt{\frac{ab+ca+b^2+c^2}{bc(a+b+c)}}\geq\sum_{cyc}\sqrt{\frac{4\sqrt[4]{a^2b^3c^3}}{bc(a+b+c)}}$$

$$=\frac{2}{\sqrt{a+b+c}}\sum_{cyc}\sqrt[8]{\frac{a^2}{bc}}\geq\frac{2}{\sqrt{a+b+c}}.3\sqrt[3]{\sqrt[8]{\frac{a^2}{bc}}.\sqrt[8]{\frac{b^2}{ca}}.\sqrt[8]{\frac{c^2}{ab}}}=\frac{6}{\sqrt{a+b+c}}.$$

Also by AM − GM inequality, we have

$$ab^2+bc^2+ca^2=\sum_{cyc}\frac{2ca^2+ab^2}{3}\geq\sum_{cyc}\sqrt[3]{(ca^2)^2.ab^2}=\sqrt[3]{(abc)^2}.(a+b+c).$$

Therefore

$$\sum_{cyc}\sqrt{\frac{1}{b}+\frac{1}{c}-\frac{2}{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{ab^2+bc^2+ca^2}}{3\sqrt[6]{abc}}\geq\frac{6}{\sqrt{a+b+c}}+\frac{\sqrt{6}.\sqrt[4]{a+b+c}}{3}$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{\frac{6}{\sqrt{a+b+c}}.\left(\frac{\sqrt{6}.\sqrt[4]{a+b+c}}{2.3}\right)^2}=3$$

Equality holds iff $a=b=c=12$.