ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, *then* :

$$\left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2 + 2\sum_{cyc} a\left(\frac{1}{b}+\frac{1}{c}\right) \geq 15$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0$,
$y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius
$= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1)$
$\Rightarrow a = s-x, b = s-y, c = s-z \therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow$
$\sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3)$

$$\therefore \left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2 + 2\sum_{cyc} a\left(\frac{1}{b}+\frac{1}{c}\right) \geq 15$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} \frac{243r^4s^2}{s^2(4Rr+r^2)^2} + 2.\sum_{cyc}\frac{a^2(b+c)}{abc} \geq 15$$

$$\Leftrightarrow \frac{243r^2}{(4R+r)^2} + 2.\frac{\sum_{cyc}\left(ab\left(\sum_{cyc} a - c\right)\right)}{abc} \geq 15$$

$$\Leftrightarrow \frac{243r^2}{(4R+r)^2} + 2.\frac{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc}{abc} \geq 15$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} \frac{243r^2}{(4R+r)^2} + 2.\frac{s(4Rr+r^2) - 3r^2 s}{r^2 s} \geq 15$$

$$\Leftrightarrow 2(4R-2r)(4R+r)^2 + 243r^3 \geq 15r(4R+r)^2 \Leftrightarrow 8t^3 - 15t^2 - 9t + 14 \geq 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left(8(t^2-4) + t + 25\right) \geq 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \left(\frac{9\sqrt{3}abc}{(a+b+c)(ab+bc+ca)}\right)^2 + 2\sum_{cyc} a\left(\frac{1}{b}+\frac{1}{c}\right) \geq 15 \ \forall\, a, b, c > 0,$$

" = " iff $a = b = c$ (QED)