ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$(a^2+b^2+c^2)\sum_{cyc}\frac{1}{(a+b-c)^2}+\frac{81\prod_{cyc}(a+b-c)^2}{(ab+bc+ca)^3}\geq 12$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly,}\left(\sum_{cyc}ab\right)^2\geq 3abc\sum_{cyc}a=12Rrs.2s\Rightarrow\left(\sum_{cyc}ab\right)^2\geq 24Rrs^2\to(1)$$

$$\text{Now,}\,(a^2+b^2+c^2)\sum_{cyc}\frac{1}{(a+b-c)^2}+\frac{81\prod_{cyc}(a+b-c)^2}{(ab+bc+ca)^3}\geq 12$$

$$\Leftrightarrow\frac{1}{4}\left(\sum_{cyc}a^2\right)\frac{\left(\sum_{cyc}(s-b)(s-c)\right)^2-2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)}{r^4s^2}$$

$$+\frac{81.64r^4s^2}{\left(\sum_{cyc}ab\right)^3}\geq 12$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)^3.\left((4R+r)^2-2s^2\right)+81.256r^6s^4\overset{(*)}{\geq}48r^2s^2\left(\sum_{cyc}ab\right)^3$$

$$\text{Now, via Trucht, (1) and }\because 3\left(\sum_{cyc}ab\right)\leq 4s^2,\text{ LHS of }(*)-\text{RHS of }(*)$$

$$\geq\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right).24Rrs^4+81.256r^6s^4-16r^2s^2.4s^2\left(\sum_{cyc}ab\right)^2\overset{?}{\geq}0$$

$$\Leftrightarrow 3R(s^2+4Rr+r^2)(s^2-4Rr-r^2)+1296r^5-4r(s^2+4Rr+r^2)^2\overset{?}{\geq}0$$

$$\Leftrightarrow(3R-4r)s^4-rs^2(32Rr+8r^2)-r^2(48R^3+88R^2r+35Rr^2-1292r^3)\overset{?}{\underset{(**)}{\geq}}0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\geq}(3R-4r)(16Rr-5r^2)s^2-rs^2(32Rr+8r^2)$$

$$-r^2(48R^3+88R^2r+35Rr^2-1292r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow(48R^2-111Rr+12r^2)s^2-r(48R^3+88R^2r+35Rr^2-1292r^3)\overset{?}{\underset{(***)}{\geq}}0$$

$$\boxed{\text{Case 1}}\;48R^2-111Rr+12r^2\geq 0\text{ and then : LHS of }(***)\overset{\text{Gerretsen}}{\geq}$$

$$(48R^2-111Rr+12r^2)(16Rr-5r^2)-r(48R^3+88R^2r+35Rr^2-1292r^3)\overset{?}{\geq}0$$

$$\Leftrightarrow 90t^3-263t^2+89t+154\overset{?}{\geq}0\;\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(90t+97)+117\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{Case\ 2}$ $48R^2 - 111Rr + 12r^2 < 0$ and then : LHS of $(***) =$

$$-\left(-\left(48R^2 - 111Rr + 12r^2\right)\right)s^2 - r\left(48R^3 + 88R^2r + 35Rr^2 - 1292r^3\right) \overset{\text{Gerretsen}}{\geq}$$

$$-\left(-\left(48R^2 - 111Rr + 12r^2\right)\right)\left(4R^2 + 4Rr + 3r^2\right)$$

$$-r\left(48R^3 + 88R^2r + 35Rr^2 - 1292r^3\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 48t^4 - 75t^3 - 85t^2 - 80t + 332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)\left(48t^2 + 117t + 191\right) + 216\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (***)$ is true $\therefore$ combining both cases, $(***) \Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore \left(a^2 + b^2 + c^2\right)\sum_{\text{cyc}} \frac{1}{(a+b-c)^2} + \frac{81\prod_{\text{cyc}}(a+b-c)^2}{(ab+bc+ca)^3} \geq 12$$

$\forall\ \Delta\ ABC$, " = " iff $\Delta\ ABC$ is equilateral (QED)